Neutrinos

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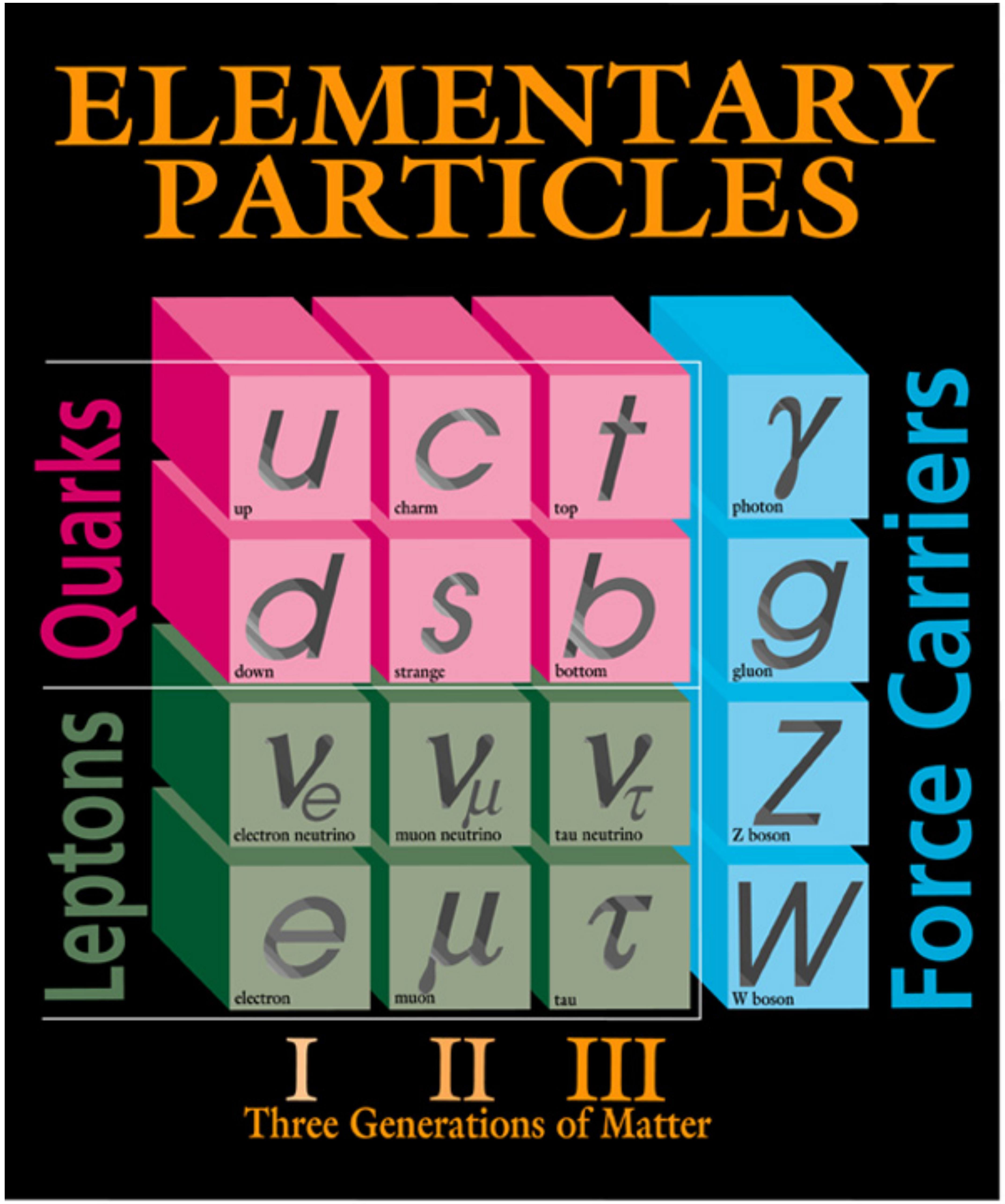
Quick Reminder

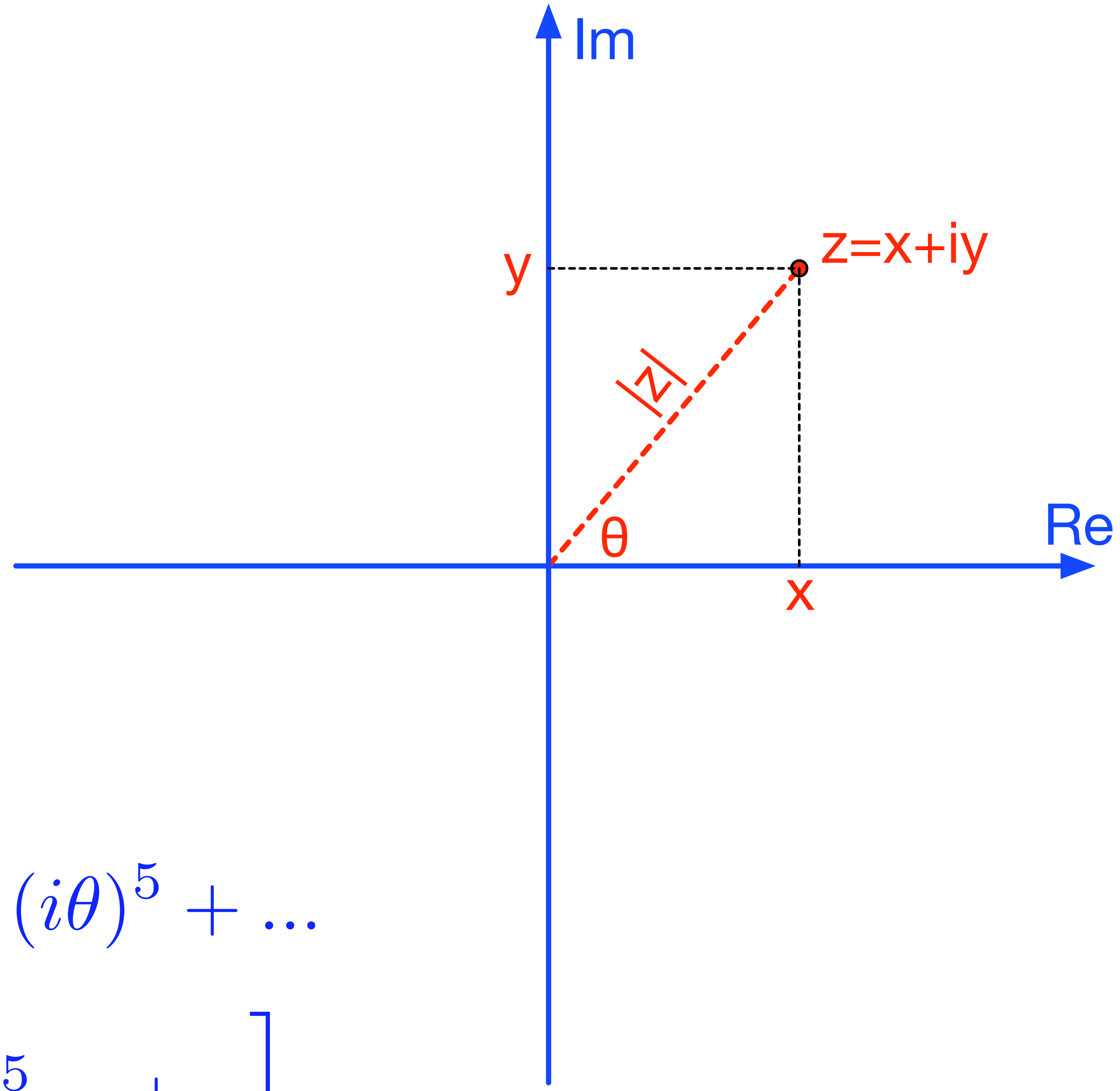
The universe is composed of three families of quarks and leptons. Today we talk about the neutral leptons, the neutrinos



Imaginary and Complex Numbers

We need a bit of quantum mechanics to understand neutrinos and we need to use complex numbers to write quantum states/operators.

Define: $i = \sqrt{-1} \quad \rightarrow \quad i^2 = -1$



A complex number has real and imaginary parts:

$$z = x + iy = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

$$z^* = x - iy = |z|(\cos\theta - i\sin\theta) = |z|e^{-i\theta}$$

$$zz^* = z^*z = x^2 + y^2 = |z|^2 e^{i\theta} e^{-i\theta} = |z|^2$$

$$\begin{aligned} e^{i\theta} &= 1 + \frac{1}{1!}(i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots \\ &= 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - + \ldots + i\left[\frac{1}{1!}\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - + \ldots\right] \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Quantum Mechanics and Measurements

- The state of a system can be characterized by a vector in a complex Hilbert space: $|\psi\rangle$
- Each state has an adjoint or conjugate state: $|\psi\rangle^\dagger = \langle\psi|$
- There is an inner product between two states: $|\psi\rangle \text{ and } |\phi\rangle = \langle\phi|\psi\rangle$
  - States are normalized: $\langle\psi|\psi\rangle = 1$
- Dynamical variables are characterized by operators on the states: $\hat{O}$
  - in general, operators modify the states: $\hat{O}|\psi\rangle = a|\phi\rangle$
  - examples: $\hat{E} = i\hbar\frac{\partial}{\partial t}, \quad \hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

- There is a special set of eigenstates and eigenvalues for each operator

$$\hat{O}|\psi_i\rangle = o_i|\psi_i\rangle, \quad i = 1, 2, \ldots$$

  - the eigenvalues o_i are real numbers, the operators $\hat{O}$ are Hermitian
  - the eigenstates are orthonormal: $\langle\psi_j|\psi_i\rangle = \delta_{ji} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$
  - the eigenstates form a complete set: $|\phi\rangle = \sum_i a_i|\psi_i\rangle, \quad \langle\phi| = \sum_i a_i^*\langle\psi_i|$
    - ✳ a_i are complex numbers

- **Measurements of the quantity O always yield an eigenvalue**
  - the **probability** of measuring o_j, (state $|\psi_j\rangle$) is $|\langle\psi_j|\phi\rangle|^2 = |a_j|^2$
  - the average of many measurements is $\langle\phi|\hat{O}|\phi\rangle = \sum_i o_i|a_i|^2$

- The time evolution of energy eigenstates is given by the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = E|\psi\rangle \quad \rightarrow \quad |\psi(t)\rangle = e^{-iEt/\hbar}|\psi(0)\rangle$$

  - complex exponential is an efficient and compact way to express the solution

- Aside: if two operators $\hat{B}$ and $\hat{C}$ commute, they share some of the same eigenvectors

$$\hat{B}|\psi_j\rangle = b_j|\psi_j\rangle \quad \rightarrow \quad \hat{B}\hat{C}|\psi_j\rangle = \hat{C}\hat{B}|\psi_j\rangle = b_j\hat{C}|\psi_j\rangle$$

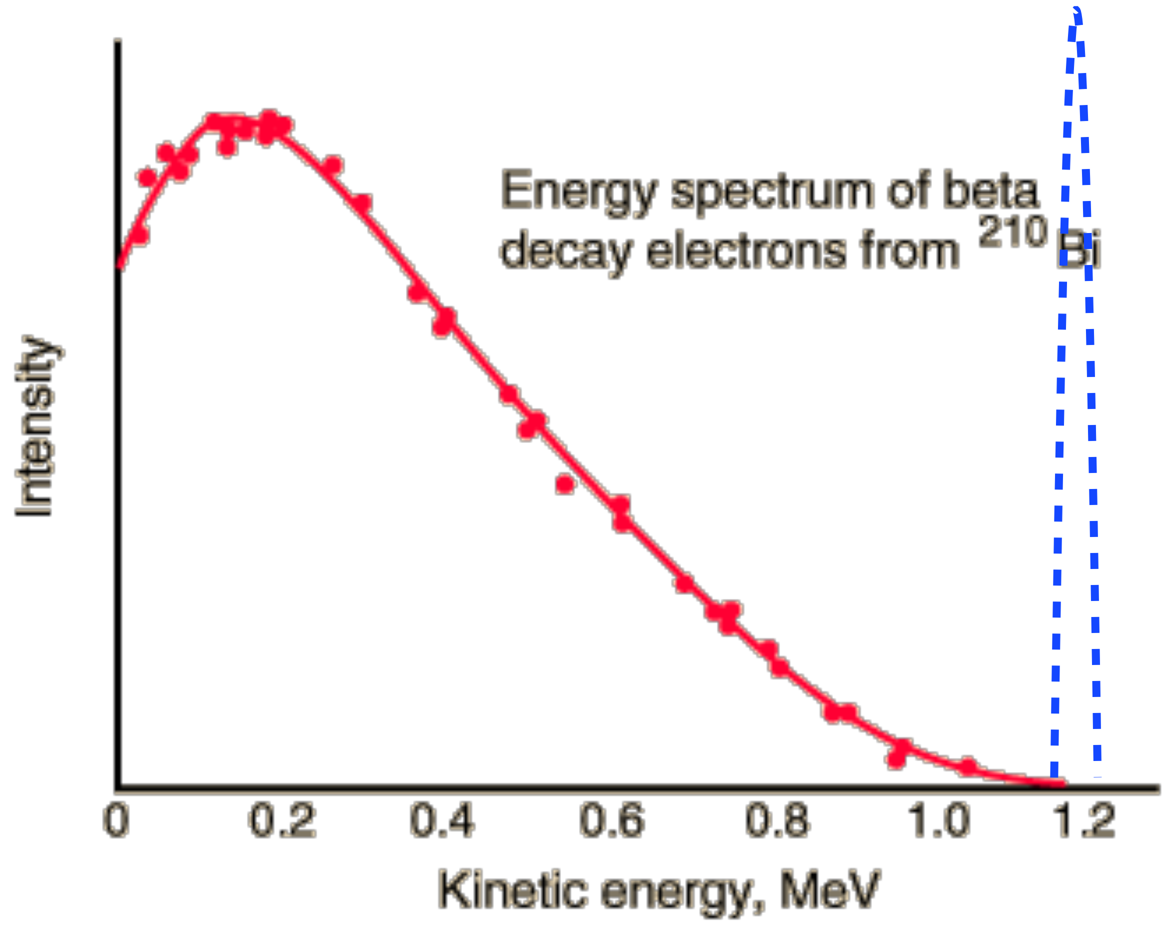
  - if the system is in an eigenstate of B it's also in an eigenstate of C: they can have definite values of of both quantities

- If they don't commute: $[\hat{B},\hat{C}] = \hat{B}\hat{C} - \hat{C}\hat{B} = \hat{D} \neq 0$, then they cannot be in definite eigenstates of both operators

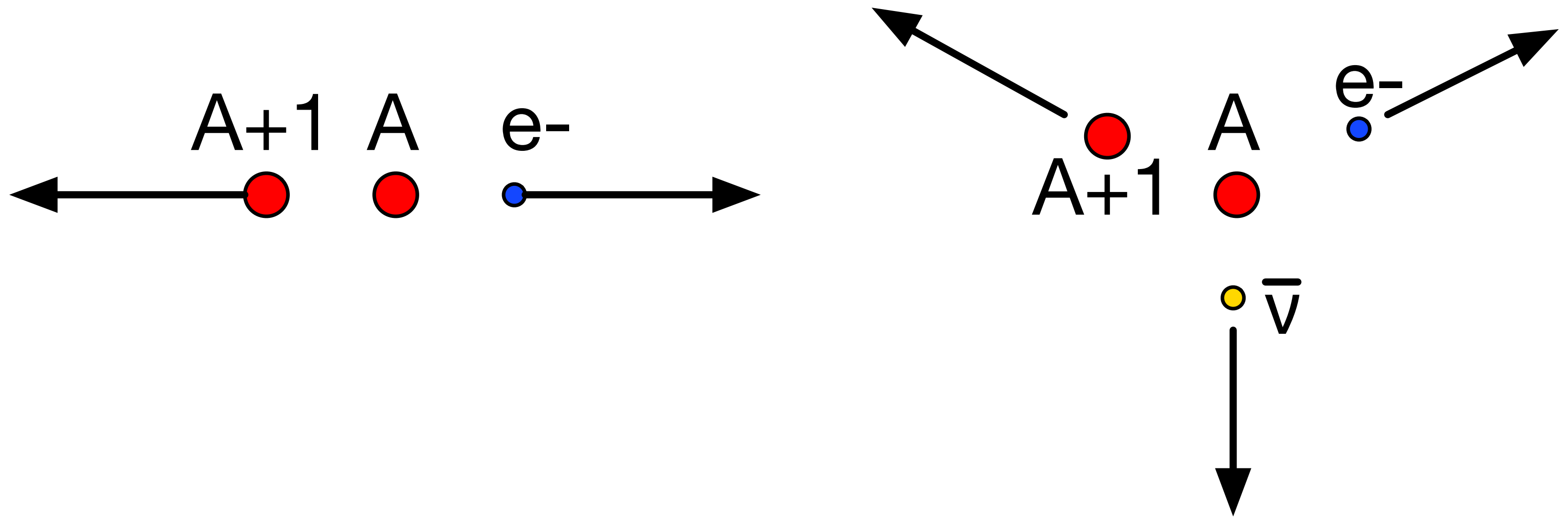
  - if we know one exactly, the other must be uncertain [Heisenberg Uncertainty Principle]

History of Neutrinos: 1930

By 1930, it was recognized that there was a problem with beta decay spectra: the electron energies were distributed in continuous spectra instead of the unique single energy of a two body decay.



- Bohr even suggested that energy conservation was only valid on average
- Pauli suggests privately in a letter to friends that there must be a light invisible particle in each decay
- he was afraid of introducing new unobserved particles
- he would not publicly present the idea until 1933



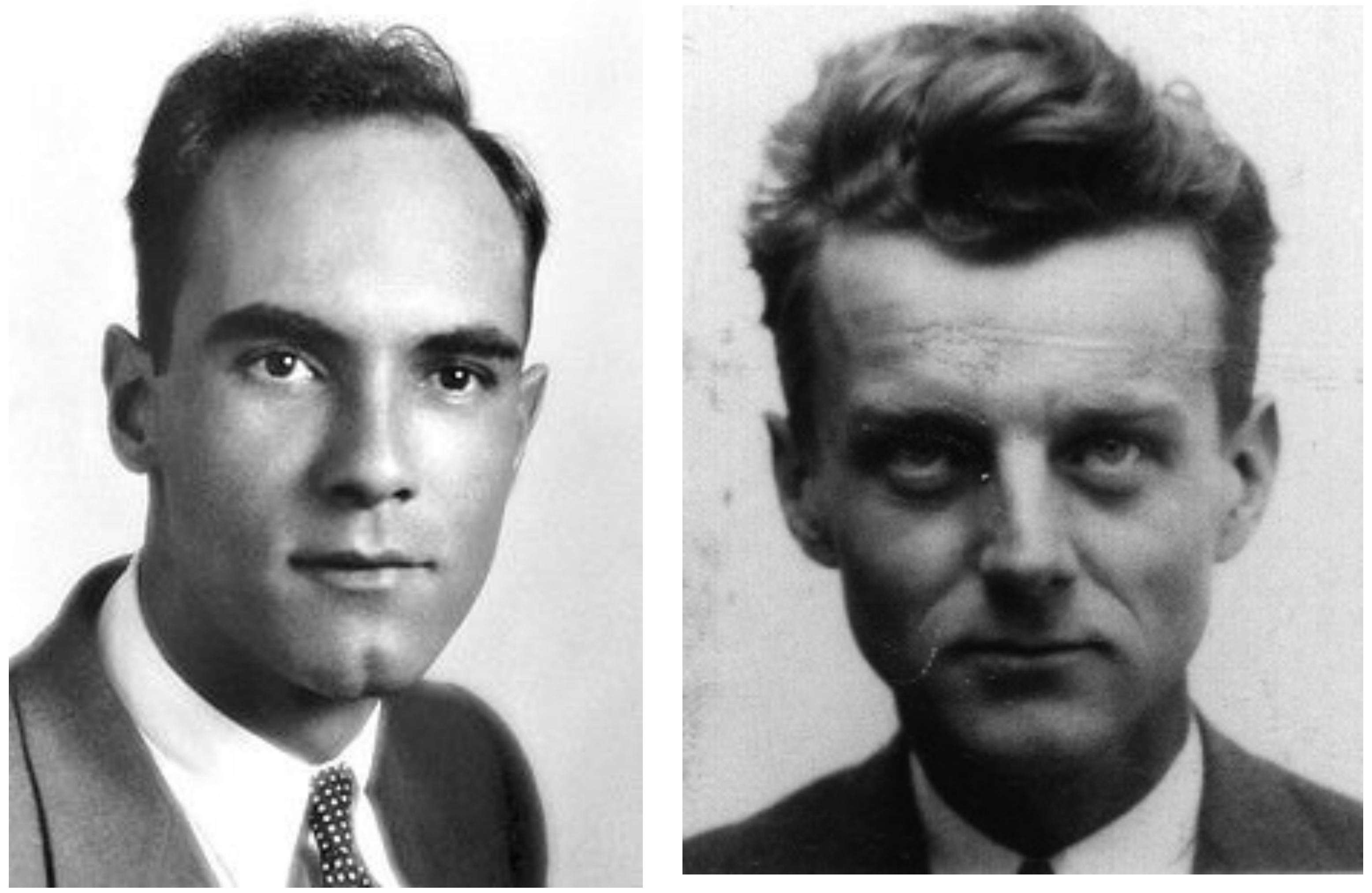
History of Neutrinos: 1933

Pauli presents his hypothesis (Solvay Conference) of a very light neutral particle to explain the continuous beta spectra. Chadwick's neutron is much too heavy and would have been detectable too.

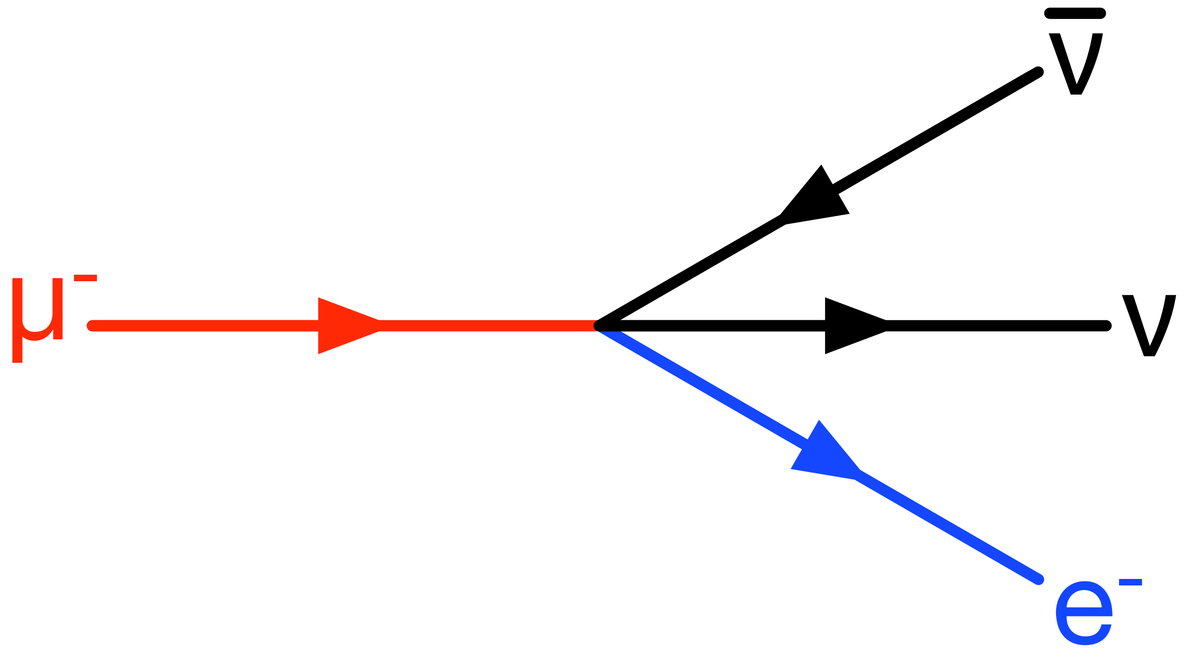
- Fermi hears Pauli's talk and within two months produces a theory of beta decay
  - would need modification in 1957 to account for parity violation
  - to distinguish the new particle from Chadwick's neutron, he names the new particle the neutrino ("little neutral one" in Italian)

History of Neutrinos: 1937

Carl Anderson and Seth Neddermeyer discover a cosmic ray particle of Q/M about 207 times smaller than the electron inside their cloud chamber!



- The mass is very close to Yukawa's predicted meson mass
  - particle is named the μ meson
  - later work would show that it was not Yukawa's meson
  - it is shown to be unstable decaying into an electron and invisible particles in 1939 with a lifetime around 2 ms

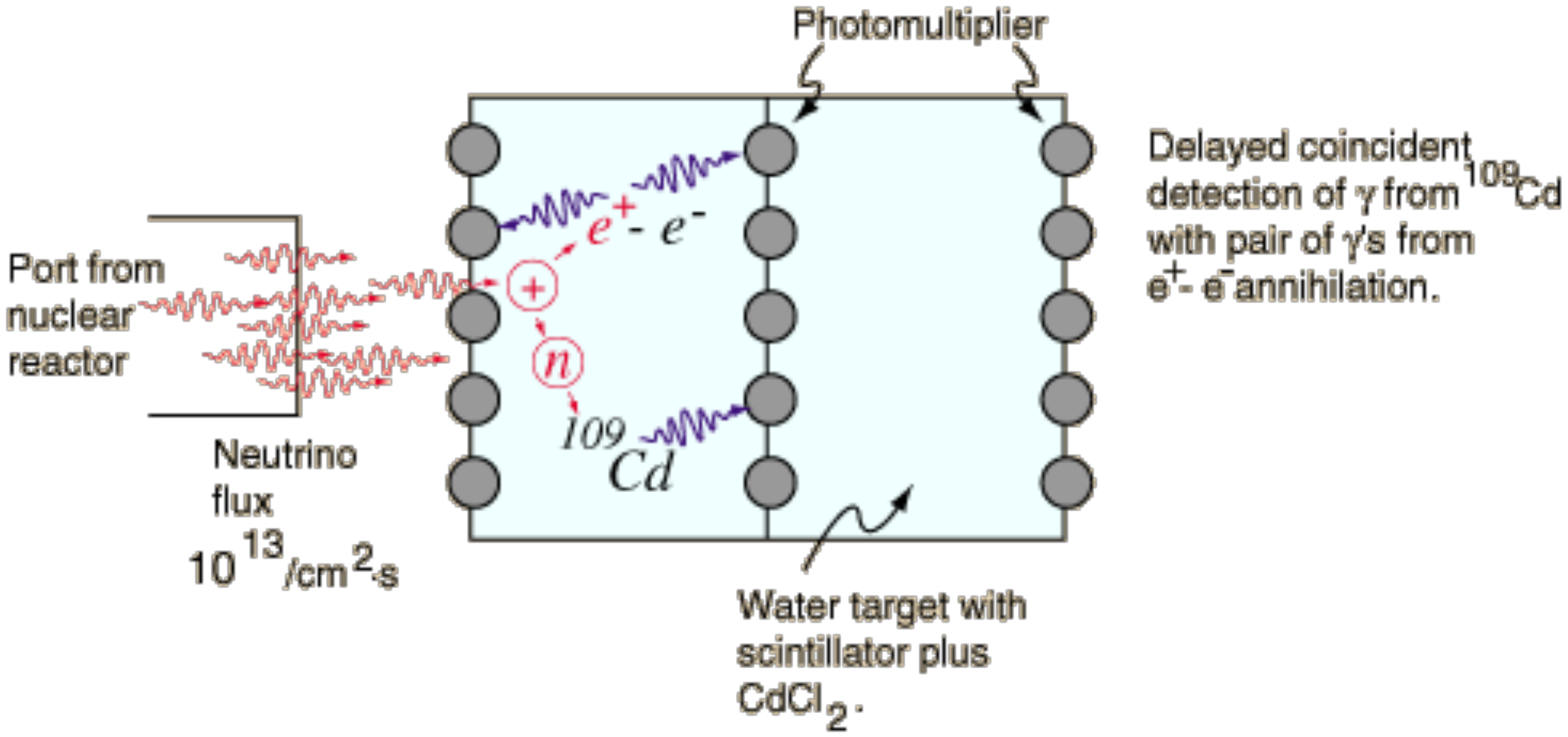


History of Neutrinos: 1956

In 1956, Cowan and Reines succeed in detecting anti-neutrinos from a nuclear reactor at Hanford WA.

$$\bar{\nu} + p \rightarrow e^{+} + n$$

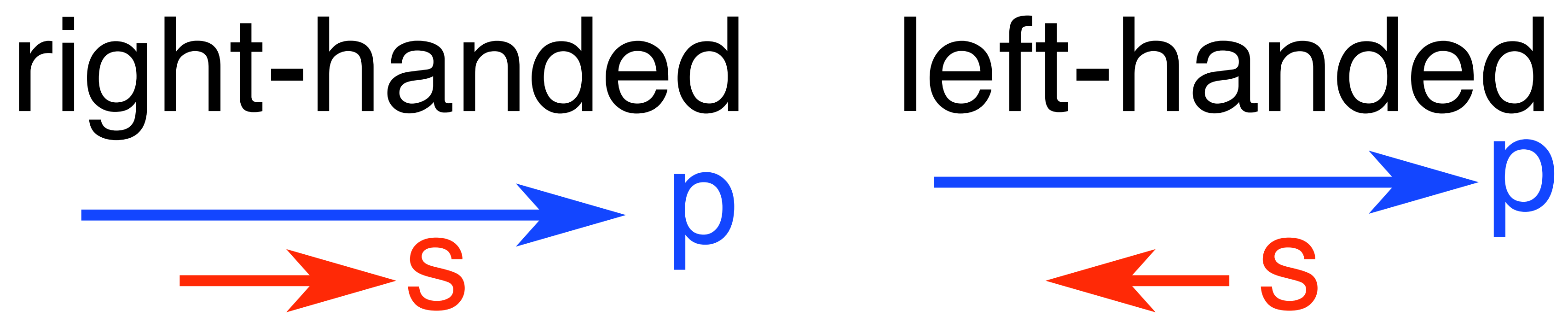
They detect photons from e^+e^- -> $\gamma\gamma$ and delayed n signal

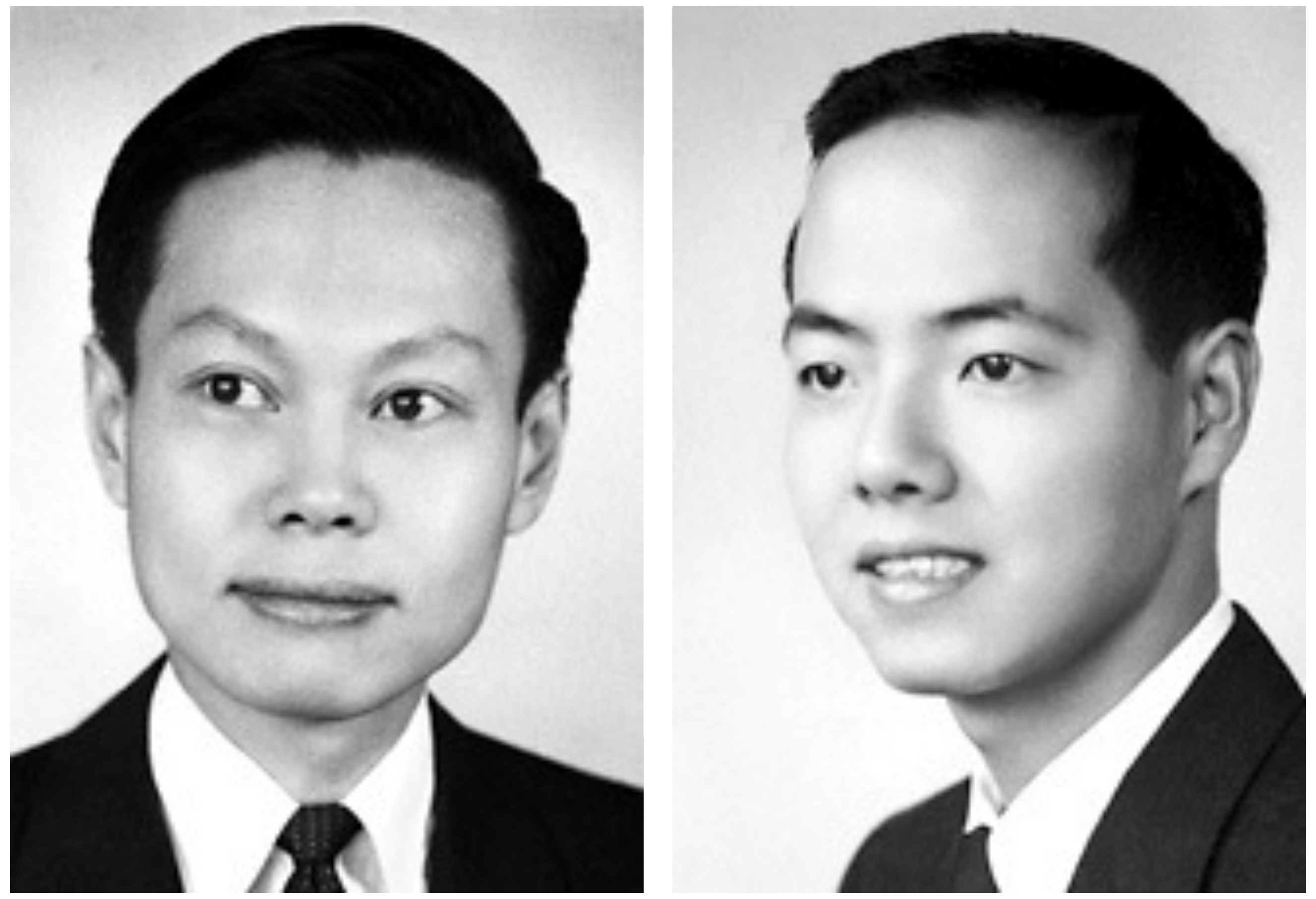


- In 1933, Pauli bet a case of champagne that no one would ever detect a neutrino,
  - he sends one to Cowan and Reines

History of Neutrinos: 1957

In 1957, T.D. Lee and C.N. Yang propose that weak interactions are not mirror-symmetric: that left-handed spinning electrons and right-handed spinning electrons behave differently.

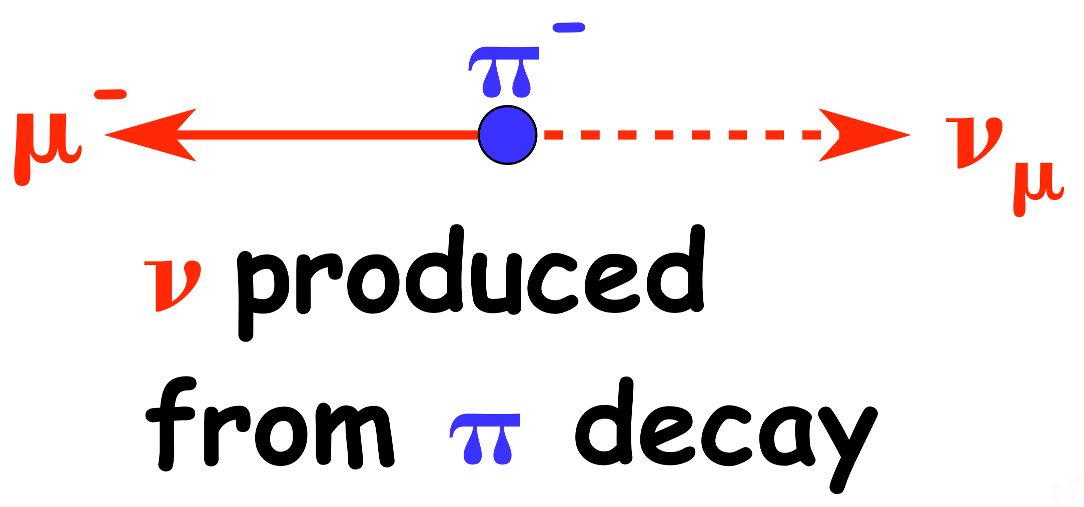


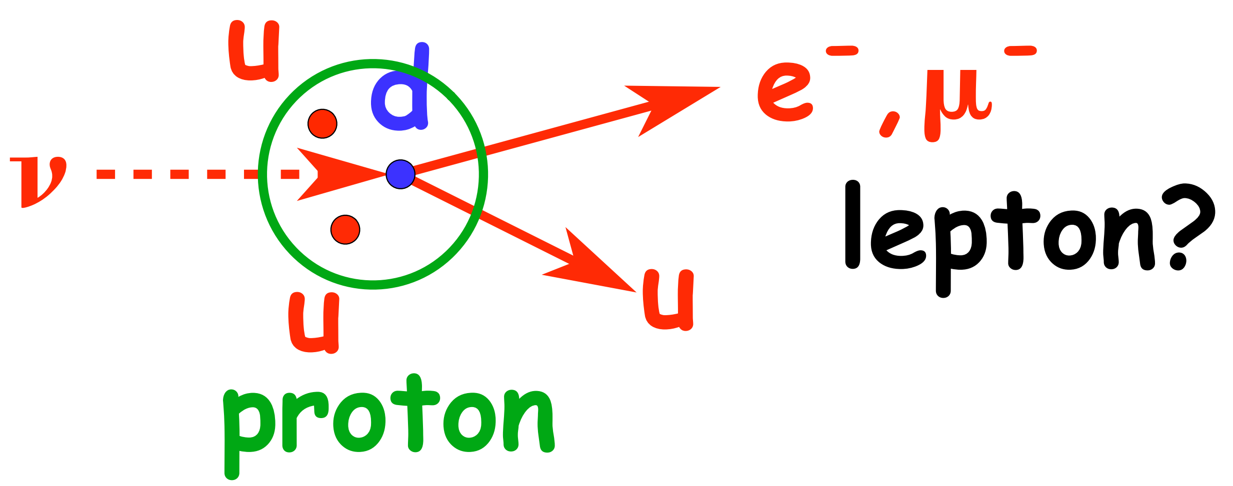


- Confirmed in a beta decay experiment by CS Wu et al
- Confirmed in two weekend experiments
  - Letterman and Garwin at BNL
  - Friedman and Telegdi in Chicago
- The weak interaction involves only left-handed particles and right-handed antiparticles
  - true for charged leptons and neutrinos

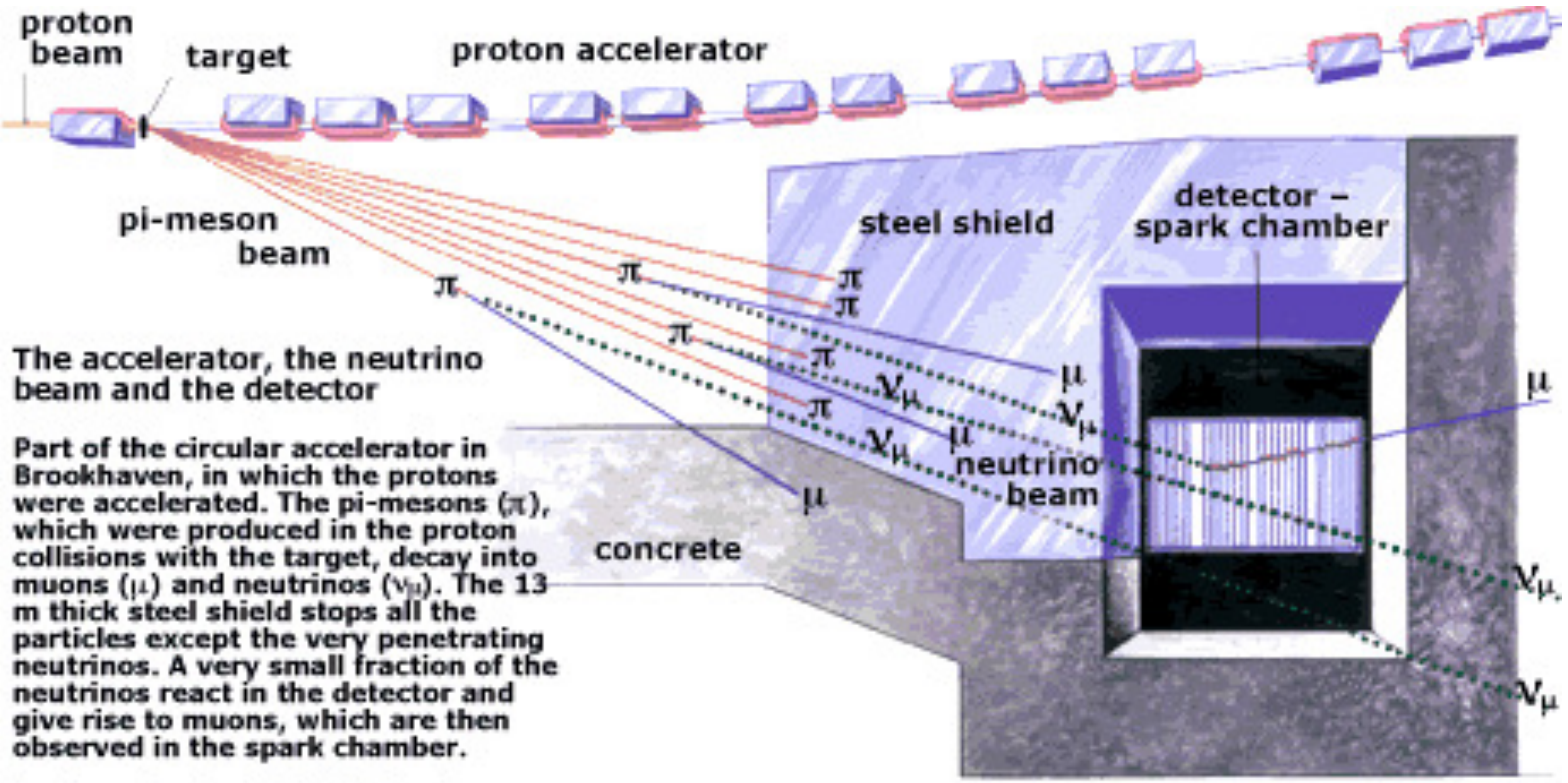
History of Neutrinos: 1962

In 1962, Lederman, Schwartz, and Steinberger did an experiment to determine if neutrinos come in e and μ flavors. They made neutrinos from the decays of pions which decay mostly into muons. If the muon were associated with its own neutrino, then those same neutrinos should preferentially produce muons when they interacted with protons/ neutrons,





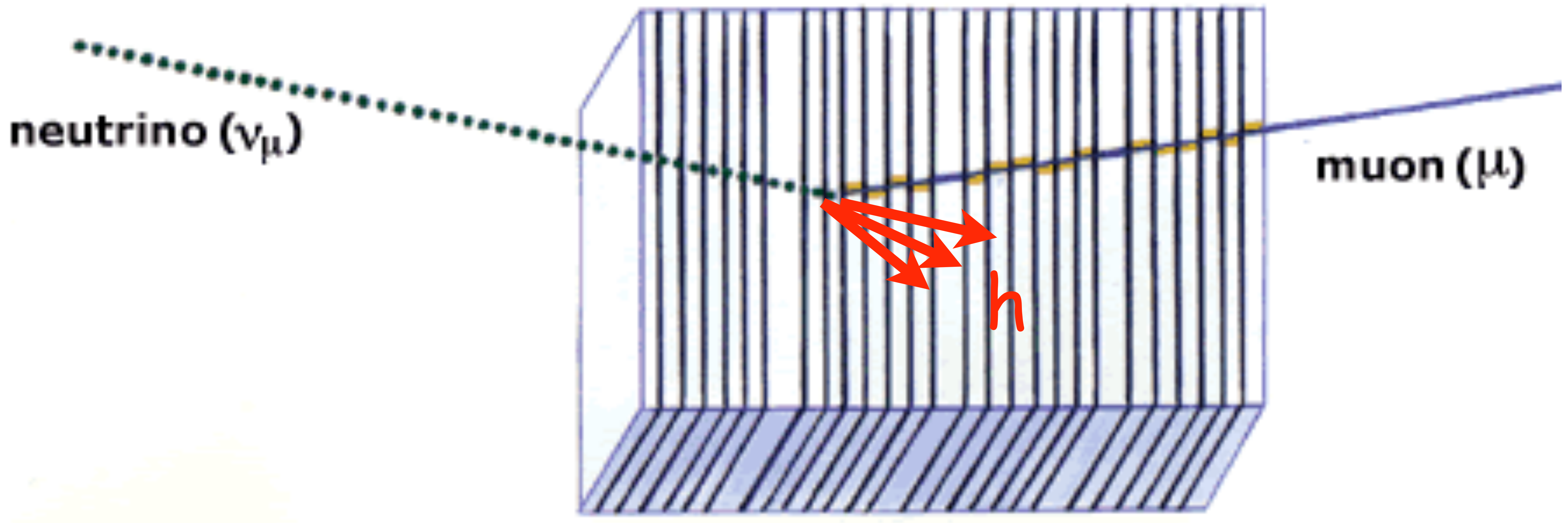
If all neutrinos are the same, then they should see both e^- and μ^- !



The accelerator, the neutrino beam and the detector

Part of the circular accelerator in Brookhaven, in which the protons were accelerated. The pi-mesons (π), which were produced in the proton collisions with the target, decay into muons (μ) and neutrinos (ν_μ). The 13 m thick steel shield stops all the particles except the very penetrating neutrinos. A very small fraction of the neutrinos react in the detector and give rise to muons, which are then observed in the spark chamber.

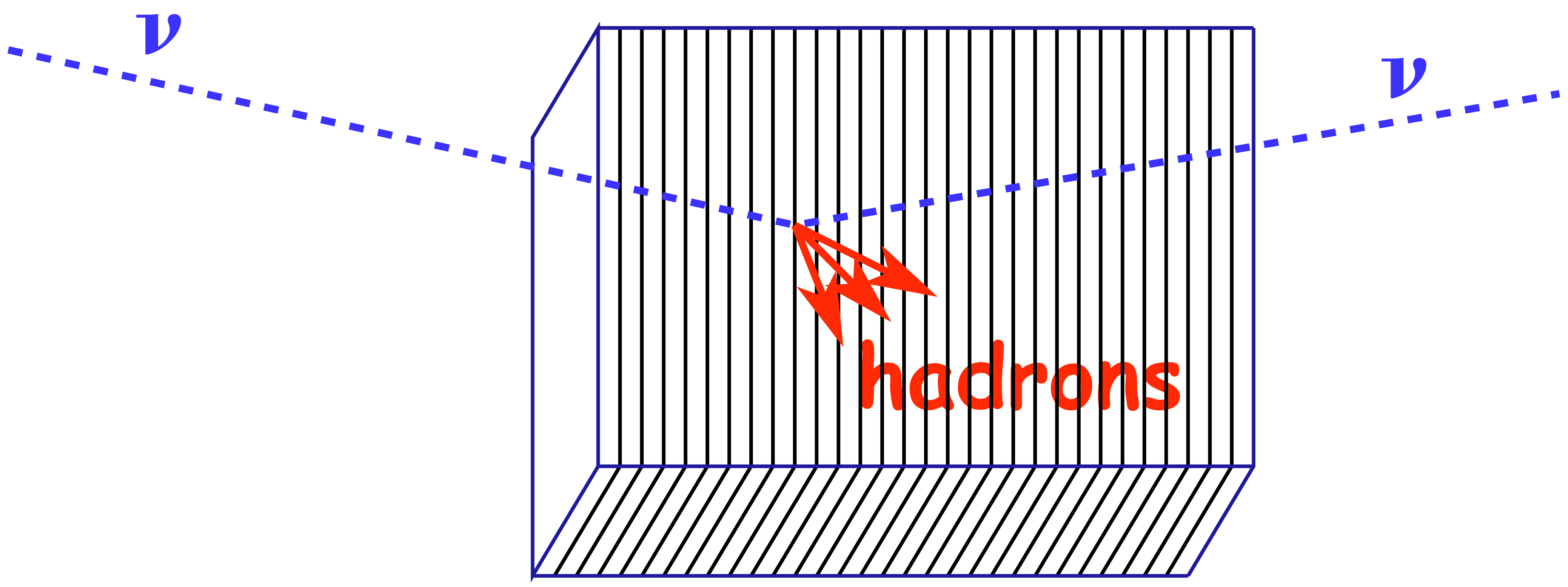
Based on a drawing in Scientific American, March 1963.



A muon produced in a neutrino reaction gives rise to discharges observed in the spark chamber.

They saw 51 events with muons coming out:

They saw zero events with an electron:
there are 2 kinds of neutrino!



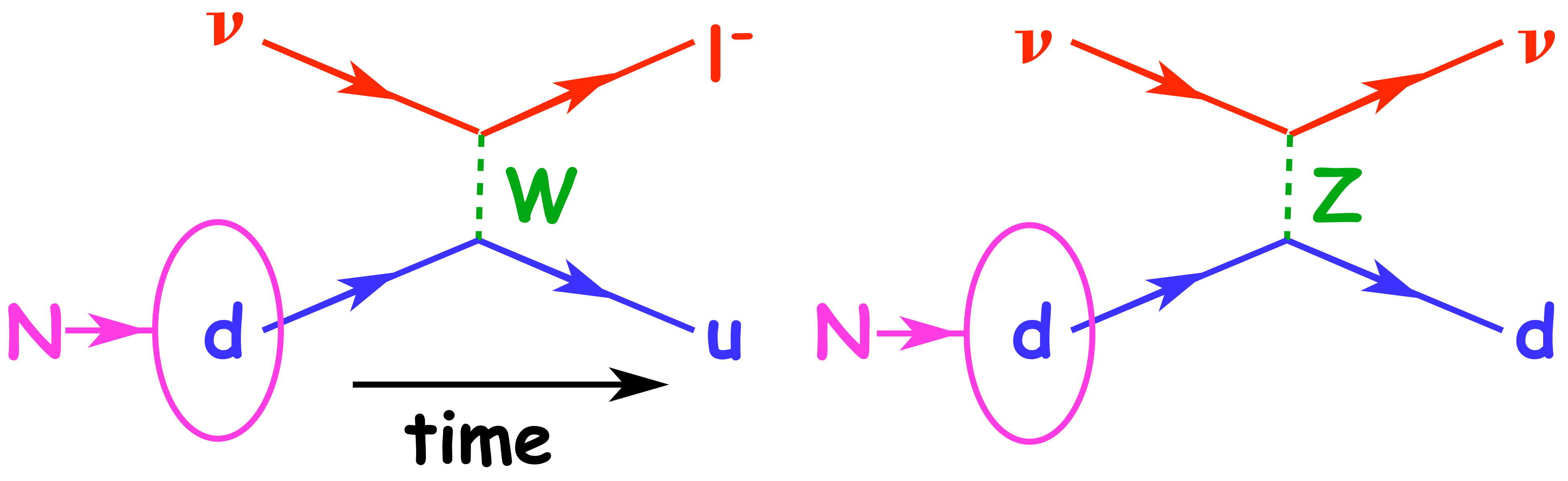
But, they also saw 7-8 events with no muon
and no electron in the final state:

- They had no idea what these could be:
    - label them as “crappers” and ignore them
- Lederman, Schwartz, and Steinberger win 1988 Nobel Prize

History of Neutrinos: 1968

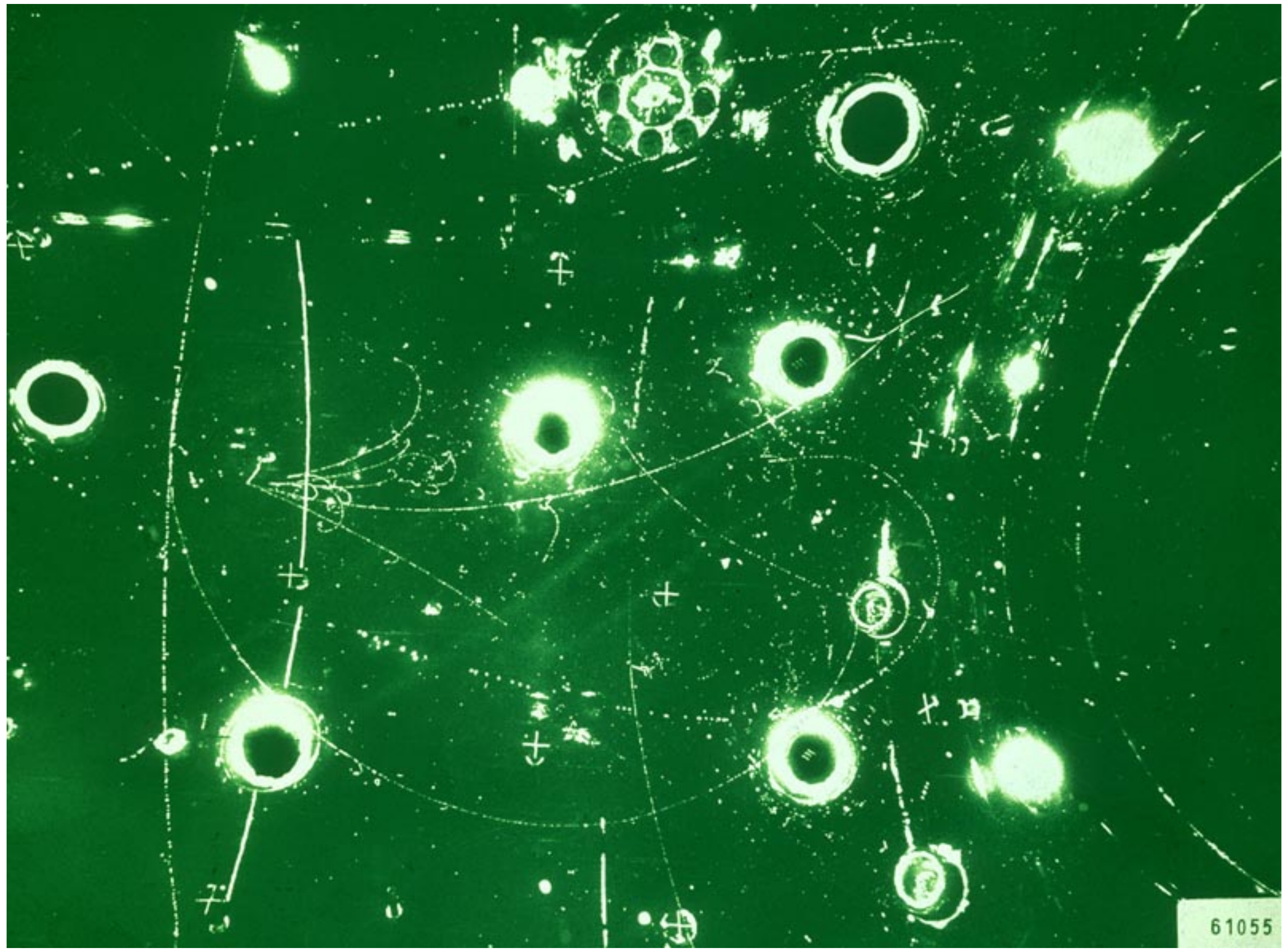
In 1968 and 1969, Weinberg and Salam independently developed a unified gauge theory of electromagnetic and weak interactions.

- massless gauge boson γ is the carrier of the electromagnetic force
- massive gauge boson W is the carrier of the weak force
  - mass limits range of force to $\hbar c/(M_W)$
- predicts a new weak neutral force mediated by Z boson
- Theory is not immediately accepted because:
  - not known to be renormalizable (later shown by Lee, Veltman, t'Huft)
  - no weak neutral force observed in nature
  - many competing theories ....



History of Neutrinos: 1973

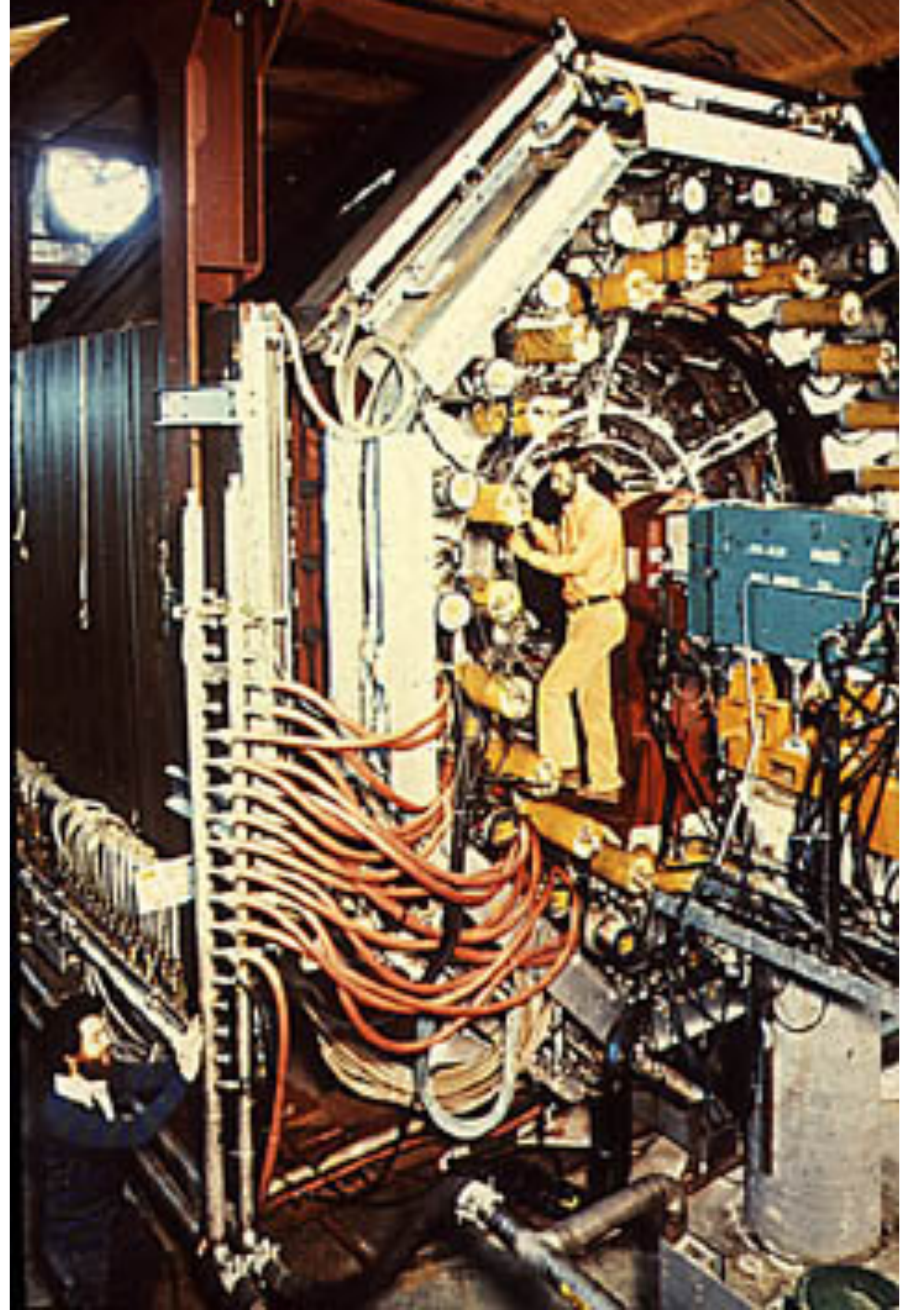
In 1973, a large heavy liquid bubble chamber Gargamelle at CERN saw more neutrino-induced events with no visible lepton in the final state



- must eliminate any backgrounds from neutrons entering the detector
    - not trivial
- A competing experiment at Fermilab, HPW, also saw evidence of neutral currents. They also had evidence against them (alternating neutral currents)

It turns out that ~1/3 of all neutrino events are neutral currents (Z-mediated scattering), but they weren't seen until there was a context to understand them!!

History of Neutrinos: 1975



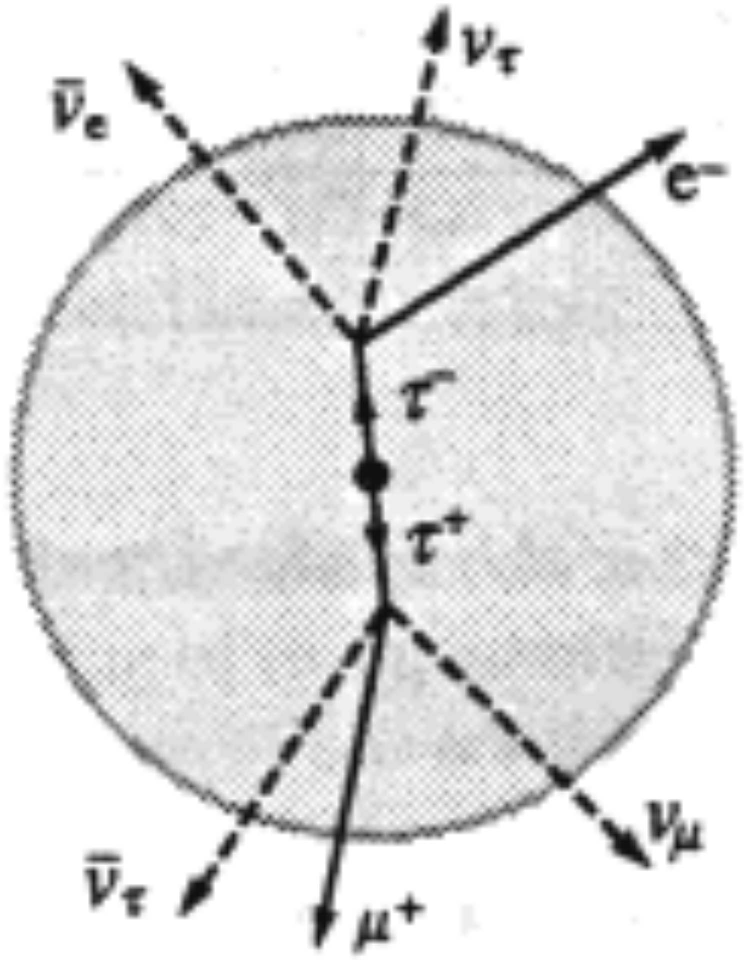
In 1975, M. Perl and the SLAC-LBL Magnetic Detector Collab observed an excess of 64 eμ events in e+e- collisions at cm energies above 4 GeV,

$$e^+e^- \rightarrow e^\pm\mu^\mp + \geq 2 \text{ undetected particles}$$

They interpreted these as the pair production of a new charged lepton of mass ~ 2 GeV

$$e^+e^- \rightarrow \tau^\pm\tau^\mp \rightarrow e^\pm\nu\bar{\nu}\mu^\mp\nu\bar{\nu}$$

- First result was a bit controversial because
    - detector covered only ~50% of the solid angle [could miss particles]
    - μ identification relied on penetration of thin iron plates [punch through?]
    - there were charmed particles being produced at similar masses
- Two collaborators refuse sign the paper
- Confirmed by subsequent work

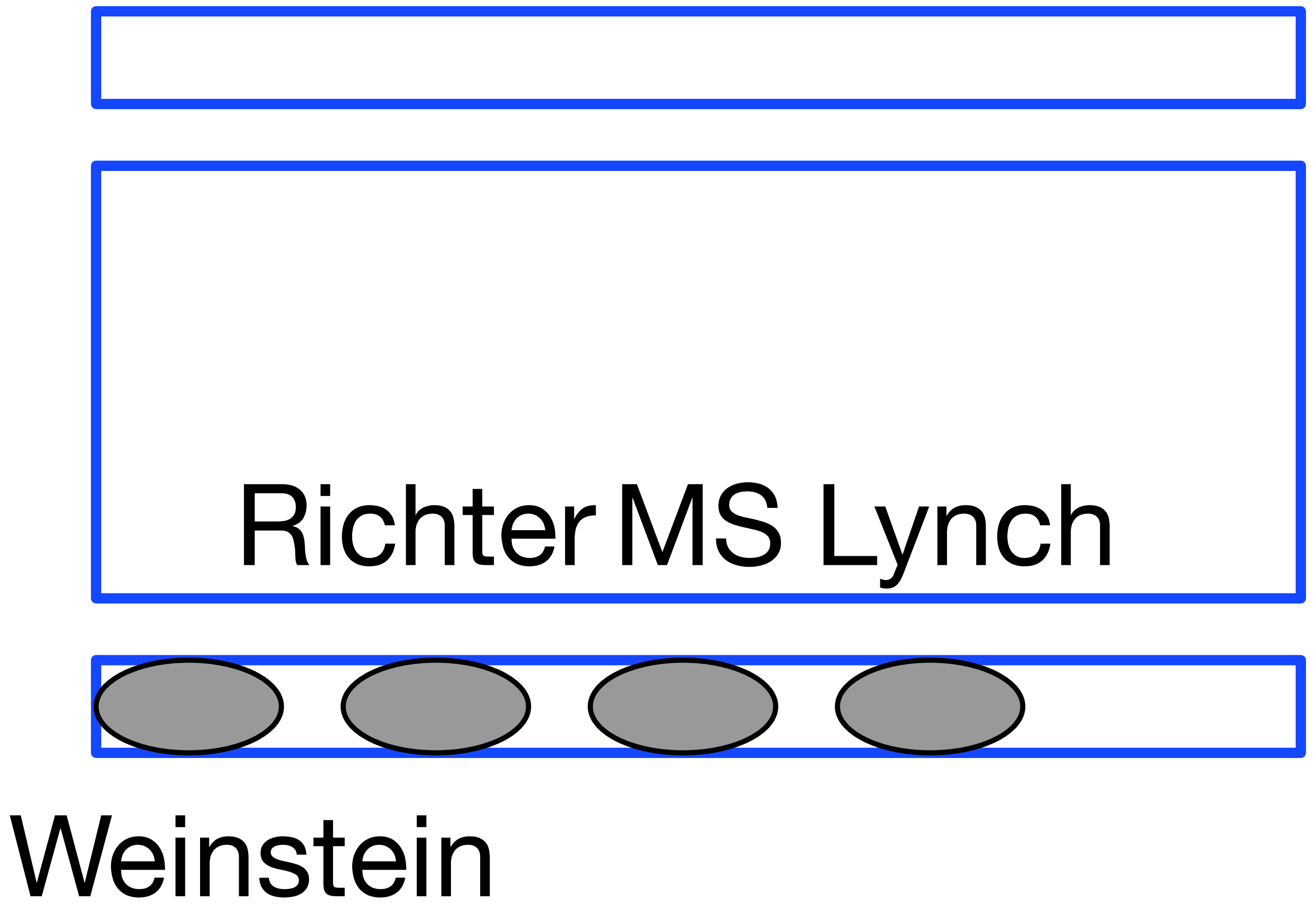


Personal aside: Lunch on 11 October 1995

Morning news: M. Perl + F. Reines share Nobel Prize for their contributions to lepton physics

Location: outdoor picnic table at SLAC
cafeteria around 12:20 PST.

- H. Lynch reiterates that the original τ lepton paper from 1975 was poorly done. He expresses satisfaction that he had refused to sign it and would not change his mind now!
    - others at the table raise eyebrows but decline to comment



- B. Richter: “This is a wonderful day for SLAC: it’s our third Nobel Prize!”
- MS: yes, it ties us with Brookhaven now.
- BR looks confused: “wait, Brookhaven had the two neutrinos expt and CP Violation, what was the third Brookhaven Nobel Prize?
- MS: “Do you remember that guy, Sam Ting, Burt?”
- BR turns bright red in the face. M. Weinstein is laughing so hard, he falls off the end of the bench and continue to laugh while lying in the grass

Standard Model of EW Interactions

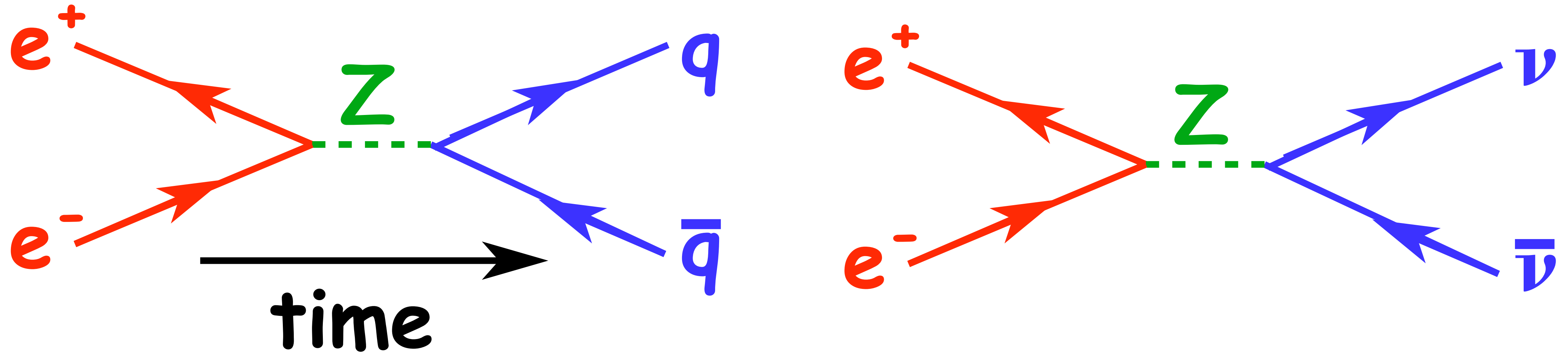
In 1982-1983, the W and Z bosons were discovered at the CERN SPS Collider. In 1989, the SLC and LEP began to produce large samples of Z bosons and to measure the detailed properties of the Z. The SM does a flawless job. It describes each of the 3 generations of fermions as consisting of 7(8) fundamental states:

- Particles come in left-chiral doublets and right-chiral singlets of weak isospin I_{3W}
  - W-bosons couple only to left-chiral doublets
- Particles have weak hypercharge $Y/2$ that couples to the B-bosons
  - electric charge $Q = I_{3W} + Y/2$
- Right-chiral neutrinos have zero weak charges
  - they are sterile: they don’t couple to gauge bosons
  - they may not exist [they aren’t really needed]

State	I_{3W}	$Y/2$	Q
u_L	1/2	1/6	2/3
d_L	-1/2	1/6	-1/3
u_R	0	2/3	-2/3
d_R	0	-1/3	-1/3
ν_L	1/2	-1/2	0
e_L	-1/2	-1/2	-1
ν_R	0	0	0
e_R	0	-1	-1

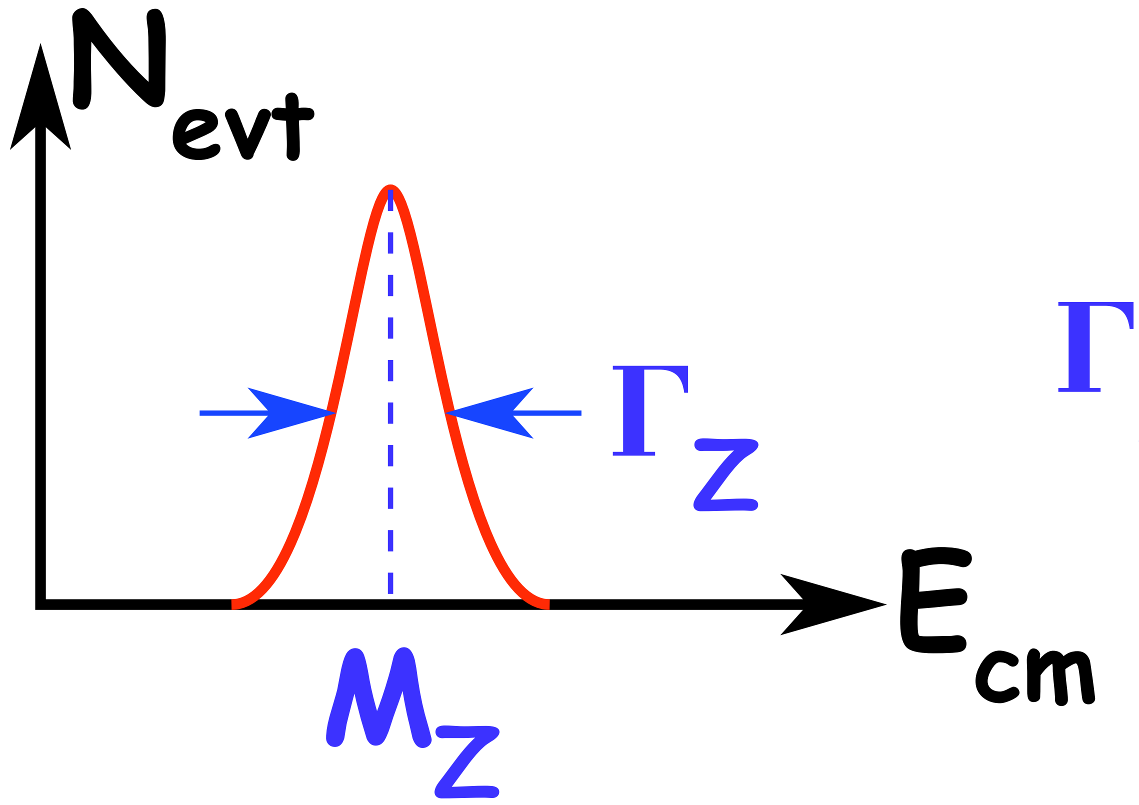
SLC/LEP Z Measurements

The e+e- -> Z -> f-fbar events produced in these machines are very clean (no background) and the energy+spin of the initial state is well-controlled



Can measure the rate of Z decays into invisible final states like v vbar by measuring Z width:

- total width measured by scanning the machine energy
  - sensitive to unseen final states like neutrinos
  - find that N_ν =2.9840±0.0082
  - confirms that there are 3 light neutrinos

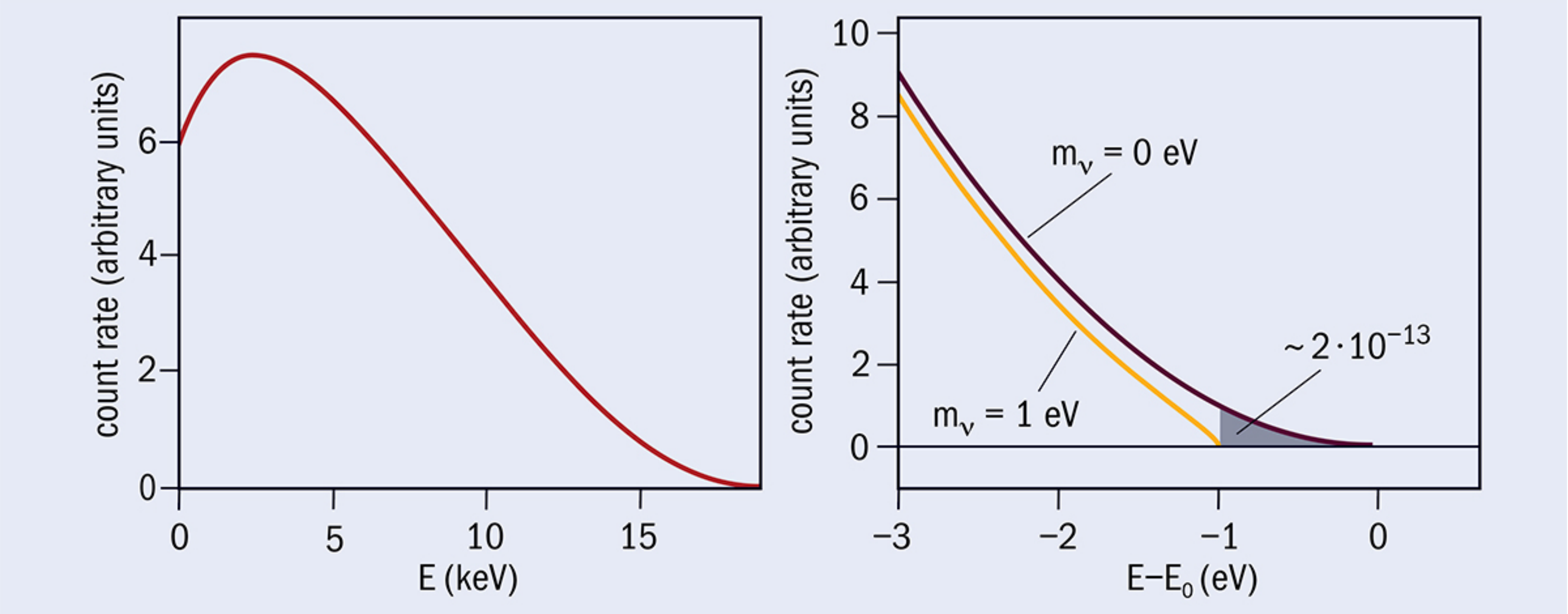


$$\Gamma_Z = \Gamma_{qq} + \Gamma_{\ell\ell} + \underset{\substack{\uparrow \\ N_\nu \Gamma_{\nu\nu}}}{\Gamma_{inv}}$$

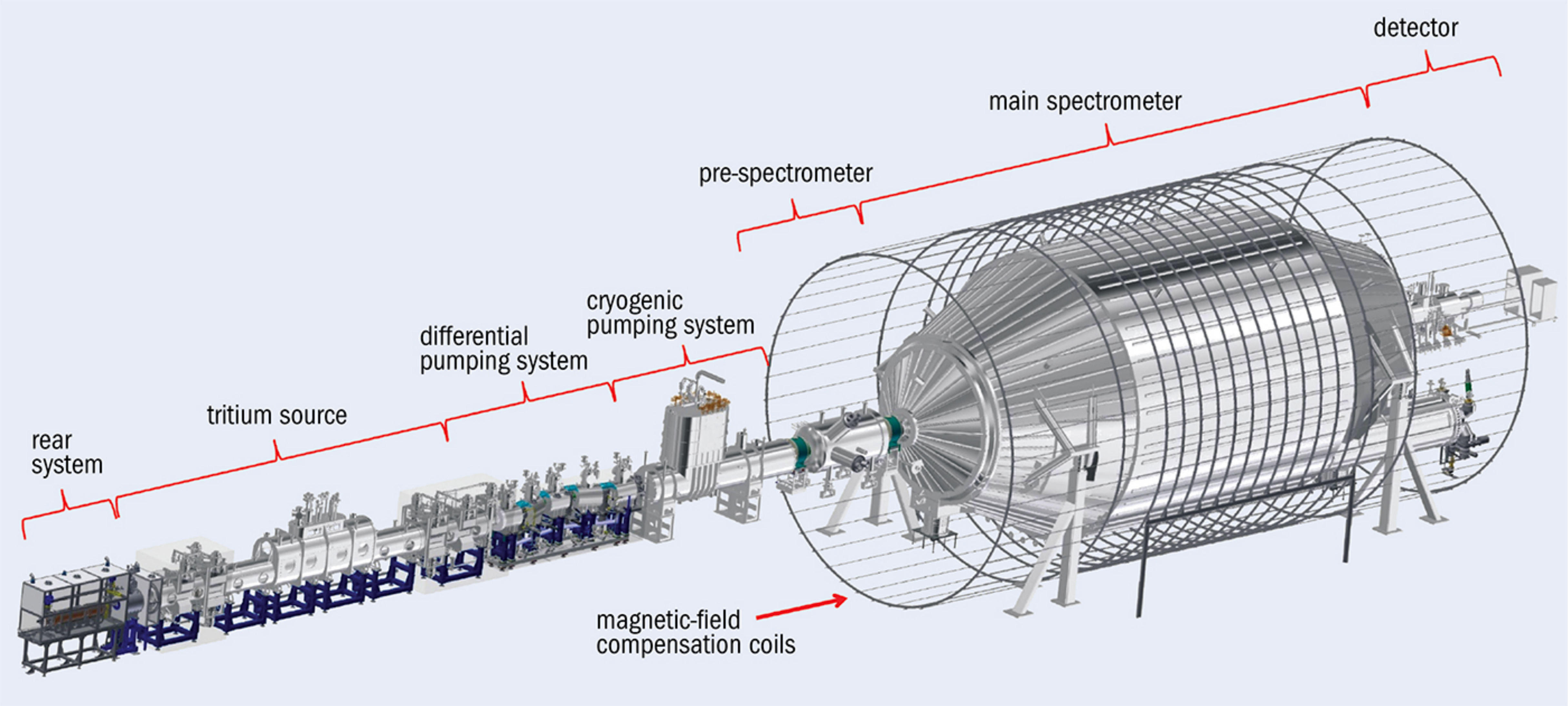
Neutrino Masses

Direct searches for neutrino mass have established that they are VERY light. The most sensitive is the measurement of the shape of tritium beta decay spectrum

$$^{3}\mathrm{H} \rightarrow {}^{3}\mathrm{He} + e^{-} + \bar{\nu}_e$$



- Smallest isotopic mass difference [18.6 KeV/c^2]
- Non-zero m_ν changes the endpoint and the shape of e^- energy distribution
  - m_ν information comes from the shape of the spectrum near the endpoint
- Best present limit from the Katrin experiment: $m_\nu < 1.1$ eV/c^2 @ 90% CL



Masses and Mixing

In the quark sector, the weak eigenstates and the mass eigenstates are not the same: they are related by the unitary CKM matrix. Similar things can happen in the neutrino sector:

- Unlike udscbt quarks, we know nothing about the neutrino mass eigenstates
  - we see only the weak eigenstates: $|e\rangle \quad |\mu\rangle \quad |\tau\rangle$
- Let's start by considering the simplest 2x2 case: we have 2 weak flavors, 2 masses and one unknown mixing angle θ

$$|e\rangle = \cos\theta|m_1\rangle + \sin\theta|m_2\rangle, \quad |\mu\rangle = -\sin\theta|m_1\rangle + \cos\theta|m_2\rangle$$

They have the correct orthonormality

$$\langle e|e\rangle = \cos^2\theta\,\underbrace{\langle m_1|m_1\rangle}_{=1} + \cos\theta\sin\theta\,\underbrace{\left[\langle m_1|m_2\rangle + \langle m_2|m_1\rangle\right]}_{=0} + \sin^2\theta\,\underbrace{\langle m_2|m_2\rangle}_{=1} = 1$$

$$\langle\mu|\mu\rangle = \sin^2\theta\langle m_1|m_1\rangle - \cos\theta\sin\theta\left[\langle m_1|m_2\rangle + \langle m_2|m_1\rangle\right] + \cos^2\theta\langle m_2|m_2\rangle = 1$$

$$\langle\mu|e\rangle = -\sin\theta\cos\theta\langle m_1|m_1\rangle + \sin^2\theta\langle m_1|m_2\rangle + \cos^2\theta\langle m_2|m_1\rangle + \sin\theta\cos\theta\langle m_2|m_2\rangle = 0$$

Let's suppose that we have an electron neutrino that is "born" at time t=0. What does it look like at later times? We've already said the energy eigenstates evolve as complex exponentials

$$|e(t)\rangle = \cos\theta e^{-iE_1 t}|m_1\rangle + \sin\theta e^{-iE_2 t}|m_2\rangle$$

The neutrino is assumed to have a definite momentum p. This means that the mass eigenstates have slightly different energies:

$$E_1 = \sqrt{p^2 + m_1^2} \simeq p + \frac{m_1^2}{2p}, \quad E_2 = \sqrt{p^2 + m_2^2} \simeq p + \frac{m_2^2}{2p}$$

This causes the two mass eigenstates to evolve with different complex phases disturbing the original mixture. The time evolved state is no longer in a weak flavor eigenstate. After time t, there is a non-zero probability that the electron neutrino has become a muon neutrino

$$\begin{aligned} P_\mu(t) = |\langle\mu|e(t)\rangle|^2 &= \left|-\sin\theta\cos\theta e^{-iE_1 t} + \sin\theta\cos\theta e^{-iE_2 t}\right|^2 \\ &= \sin^2\theta\cos^2\theta\left[e^{-iE_2 t} - e^{-iE_1 t}\right]\left[e^{iE_2 t} - e^{iE_1 t}\right] \\ &= \sin^2\theta\cos^2\theta\left[2 - \left(e^{i(E_2-E_1)t} + e^{-i(E_2-E_1)t}\right)\right] \\ &= \sin^2 2\theta\left[1 - \cos(E_2 - E_1)t\right] = \sin^2 2\theta\left[1 - \cos\left(\frac{m_2^2 - m_1^2}{2p}t\right)\right] \end{aligned}$$

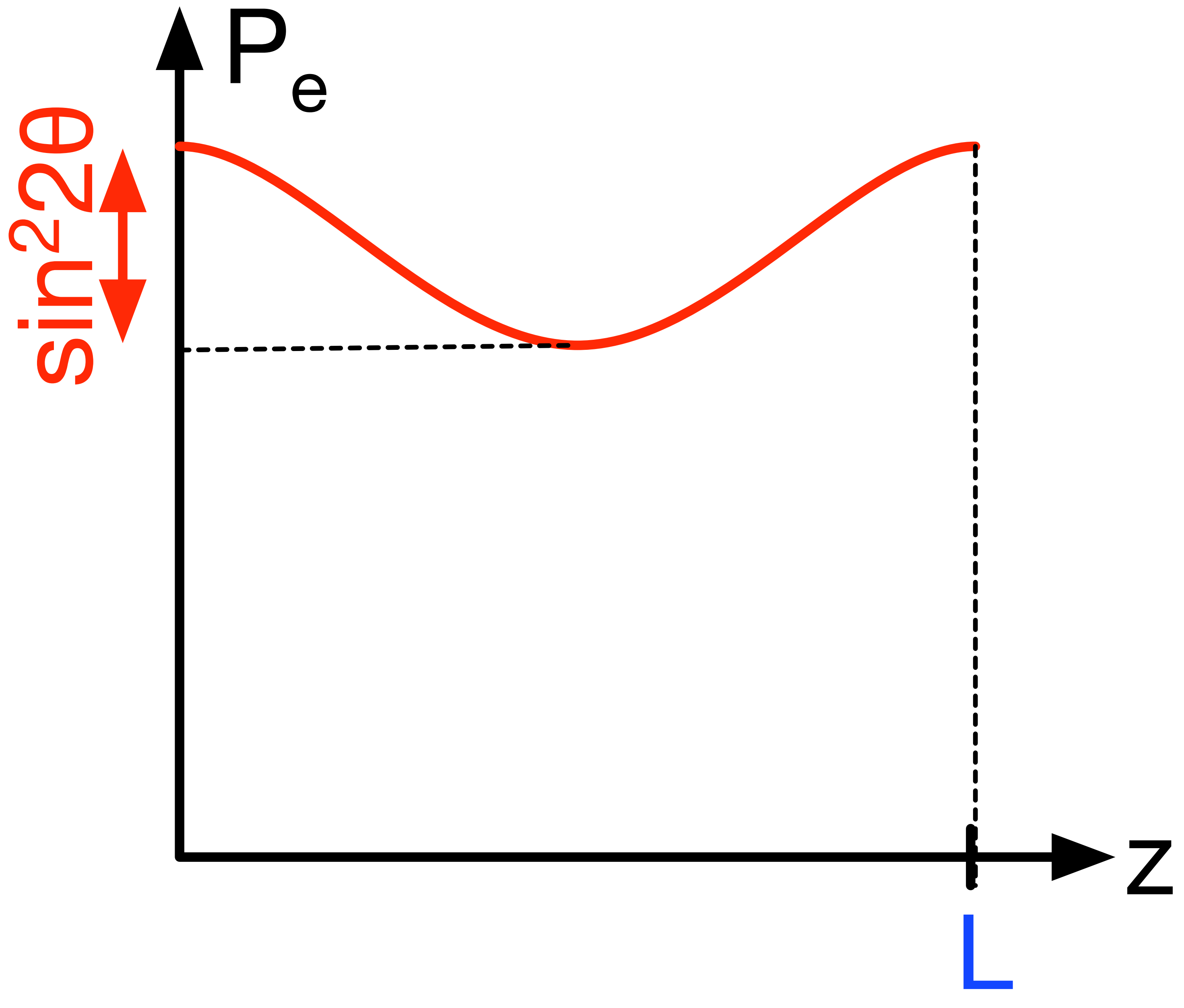
And taking $\Delta m_{21}^2 = m_2^2 - m_1^2$, $t = z/c$, $\sin^2 x = (1-\cos 2x)/2$, the transition probability is

$$P_\mu(t) = \sin^2 2\theta \sin^2\left[\frac{\Delta m_{21}^2 z}{4p}\right] \qquad P_e(t) = 1 - P_\mu(t) = 1 - \sin^2 2\theta \sin^2\left[\frac{\Delta m_{21}^2 z}{4p}\right]$$

- Neutrino oscillations were first suggested by B. Pontecorvo in 1957
  - electron neutrinos disappear, muon neutrinos appear
- The oscillation length L occurs when a full cycle is achieved:

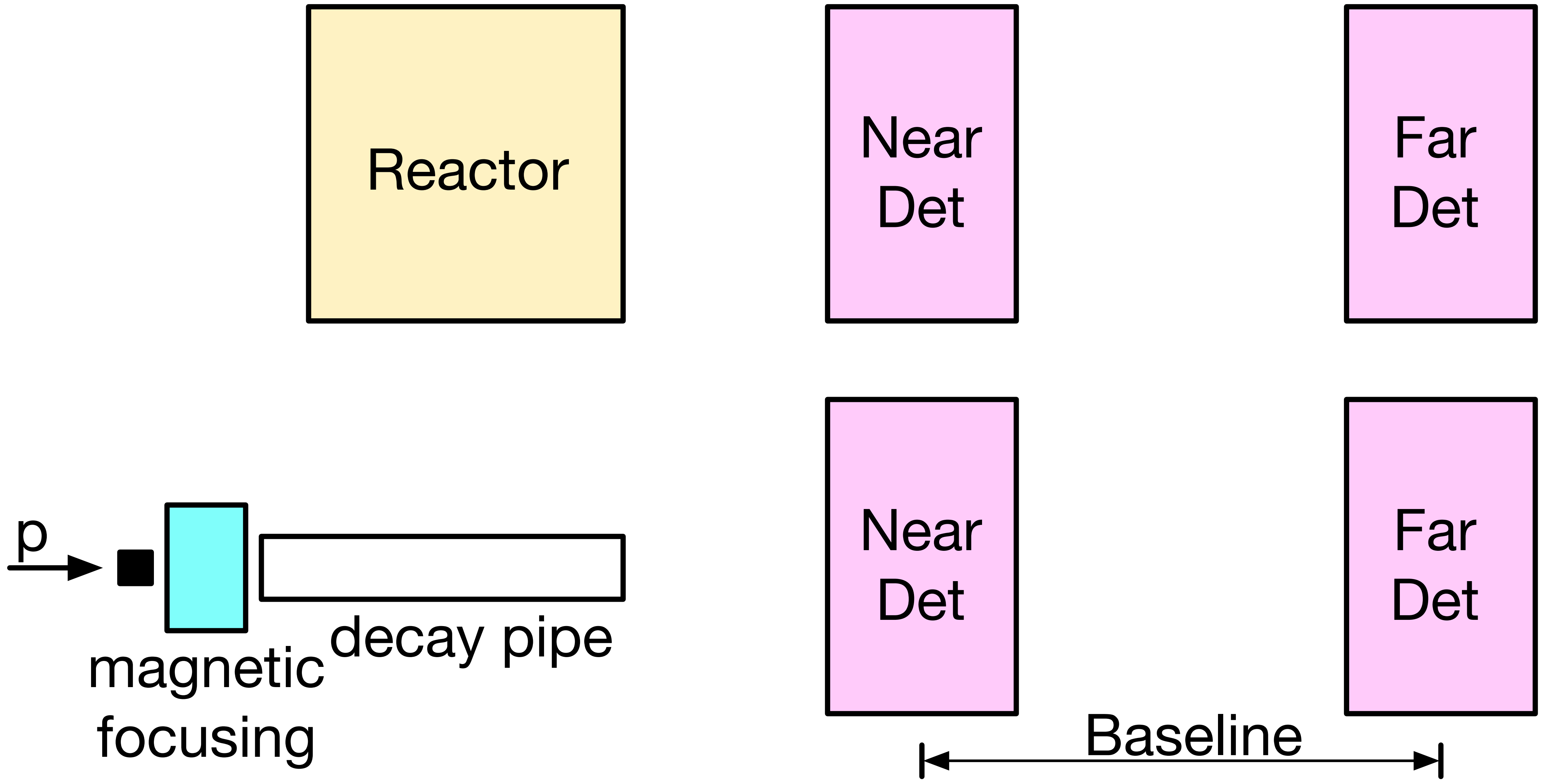
$$L = \frac{4\pi p}{\Delta m_{21}^2}$$

  - increases for small Δm_{21}^2 or large p
- amplitude of the oscillation is $\sin^2 2\theta$



Neutrino Oscillation Experiments

There have been many accelerator/reactor based searches for neutrino oscillations since the 1960s. They usually look something like this

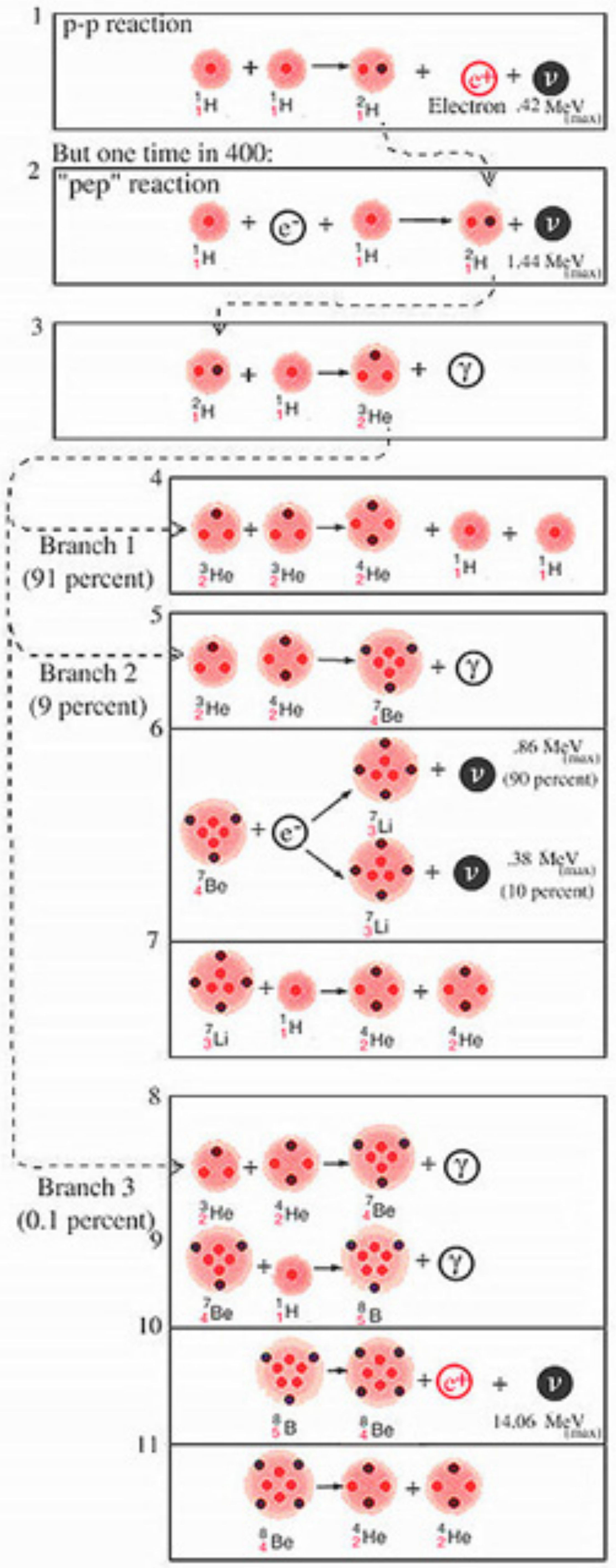


- Reactor sources produce $\bar{\nu}_e$ of 0-10 MeV
  - look for disappearance from near to far detector
- Accelerators produce (sign selected) hadron beams which decay mostly to ν_μ $[\bar{\nu}_\mu]$ with some ν_e $[\bar{\nu}_e]$ of 1-fewX10 GeV energies
  - can look for appearance or disappearance
- Prior to the 21st century, all of these produced null results
  - baselines were too short [they usually had to fit on a lab site and need very large fluxes or exposures at long distances]
- Searches using natural neutrino sources were more successful

Solar Neutrinos

There are two fusion reaction cycles in the sun: CNO and pp.
The pp produces neutrinos:

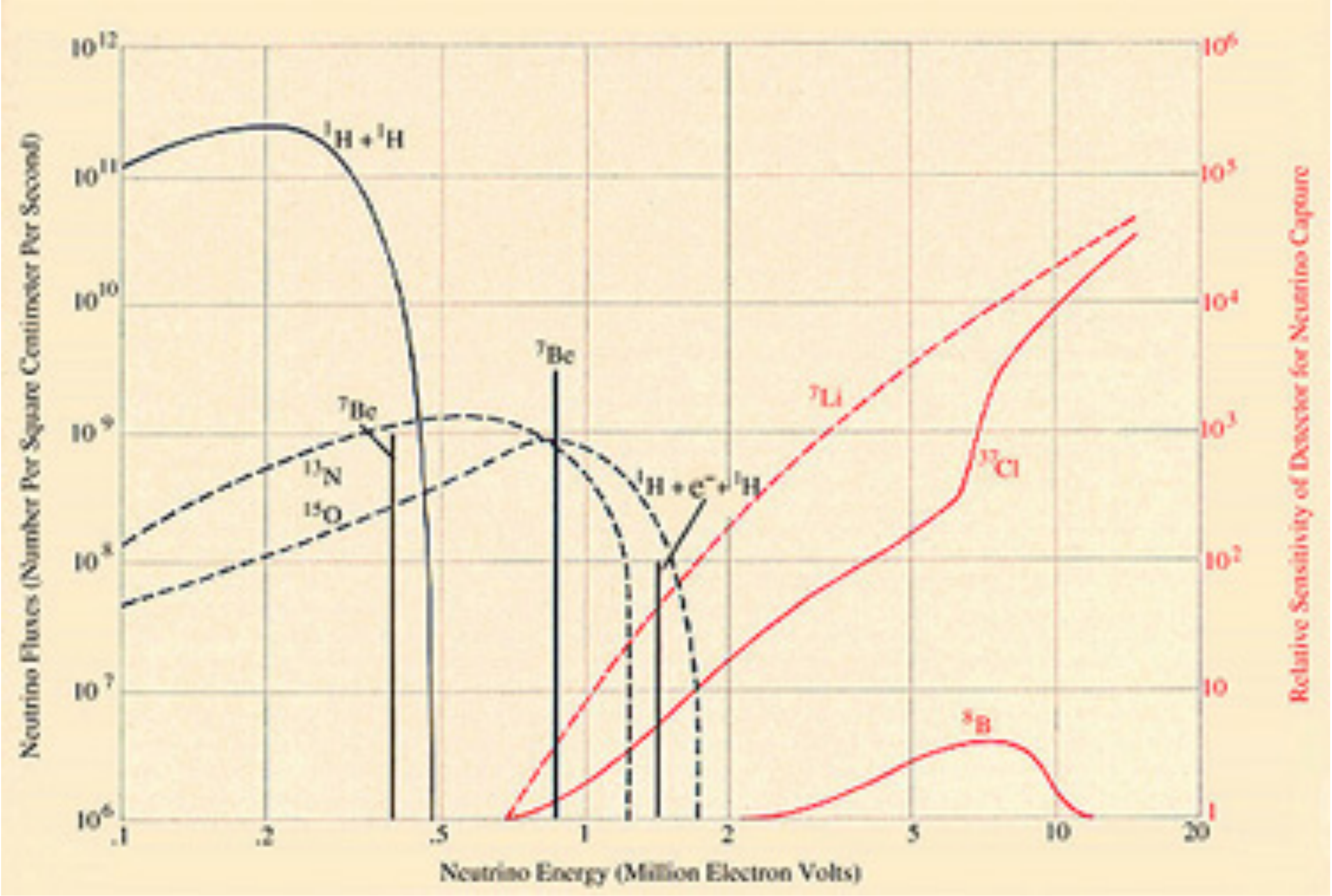
- Most of the neutrinos have sub MeV energies
    - a very few from ^{8}B decays, can have energies up to 14 MeV
- Pontecorvo suggested in 1946 that the reaction $^{37}Cl+\nu_e \rightarrow {}^{37}Ar+e^-$ could be used to detect low energy neutrinos
    - ^{37}Ar is unstable and decays back to ^{37}Cl with a 35 day half-life: it can be collected and counted in a proportional counter
    - the threshold for the reaction is 0.814 MeV, but the cross section rises steeply with ν_e energy
- In the 1965/6 Brookhaven based Chemist Ray Davis designs and builds a detector to measure the rate of neutrino emission from the sun



Homestake Neutrino Experiment

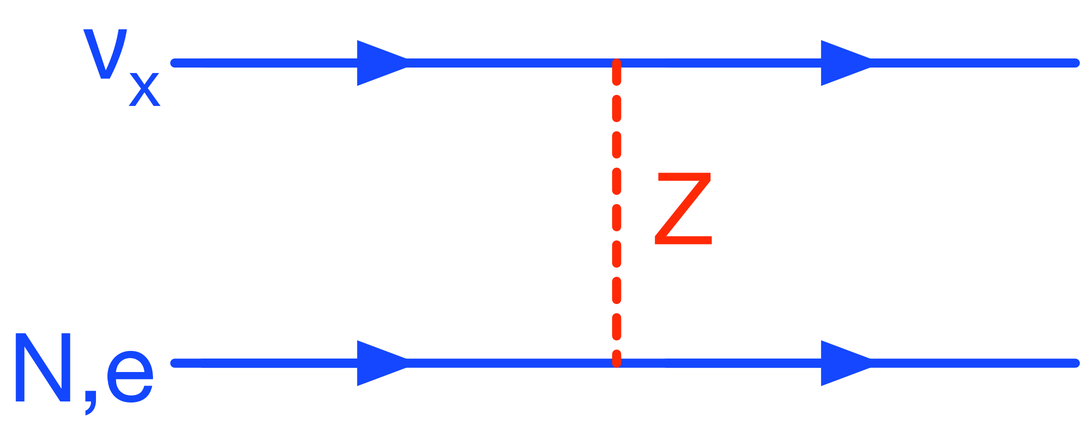
Davis and collaborators placed a 100,000 gal tank of tetrachloroethylene [C_2Cl_4] in the Homestake mine in SD.

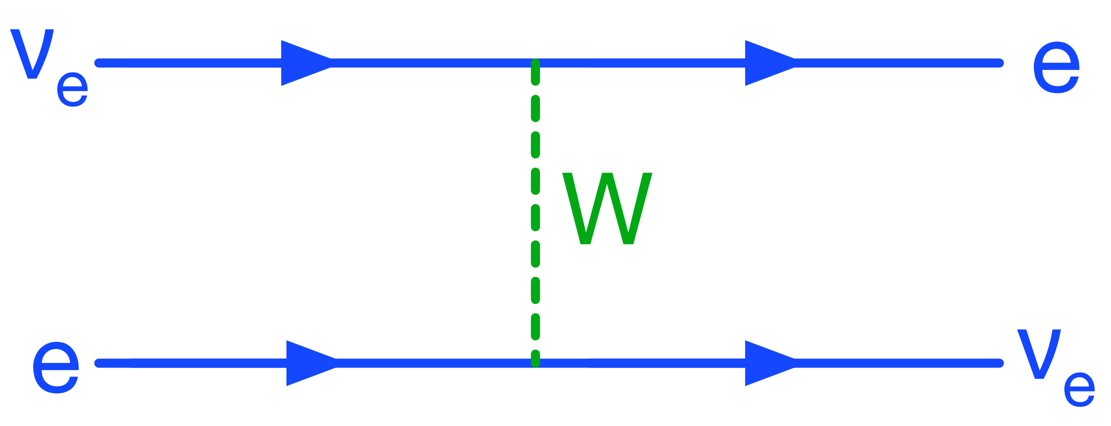
- Every few months, He is bubbled through the liquid and circulated through a cold trap froze out any remaining C_2Cl_4 and a charcoal filter
  - in 20 hours, 95% of the ^{37}Ar was removed [tested by introducing ^{36}Ar]
- The ^{37}Ar was purified and counted at Brookhaven
  - the signal was persistently around 35% of that expected from the Solar Model [from 1967 to 2001, ~2200 ^{37}Ar decays were observed]
- Other solar neutrino experiments also observed shortfalls using different target isotopes and techniques
- Much work on solar modelling established the expected flux [reducing the flux reduced the solar output too much]
- Could all of the data be explained by neutrino oscillations?



Solar Neutrino Oscillations

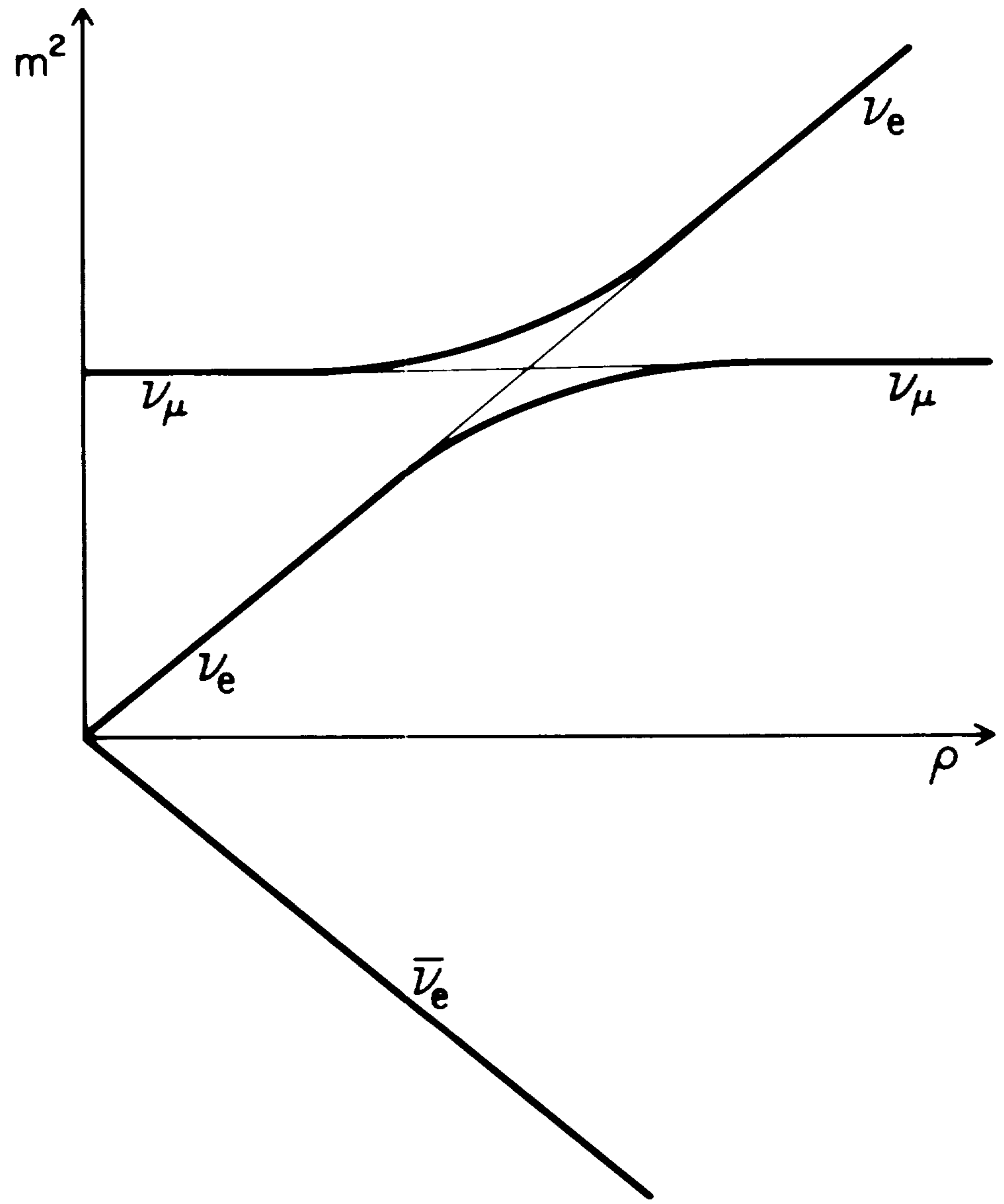
As more data came along at different neutrino energies, it became apparent that simple vacuum oscillations could not really explain the shortfall of solar neutrinos. In 1985, Mikheyev and Smirnov produced a solution using a framework developed by Wolfenstein. It had some mistakes, but they were fixed by H. Bethe. Neutrinos interact with the matter they are traversing:





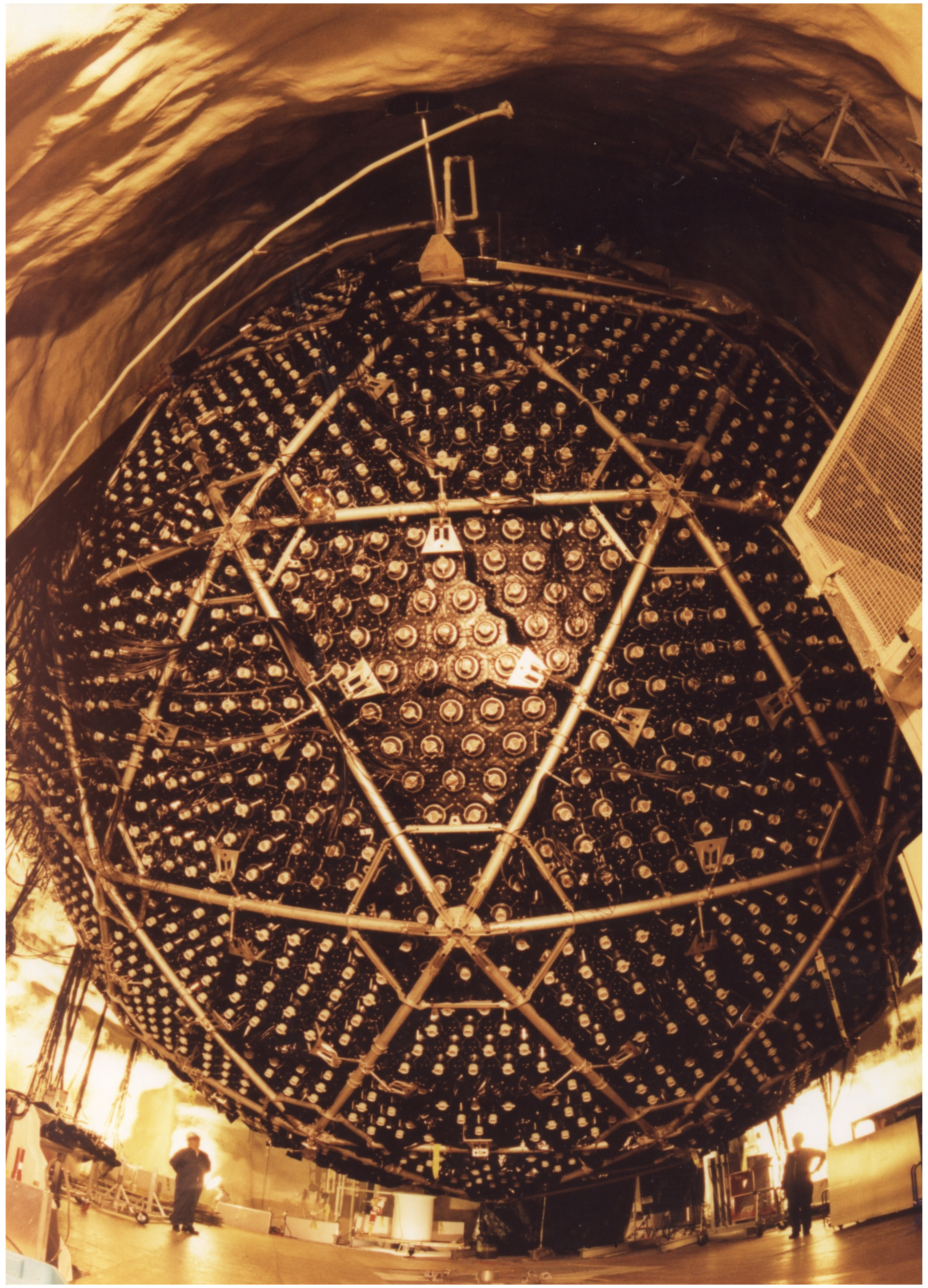
$$V_{\text{eff}} \simeq G_F\sqrt{2}n_e$$

- All ν species "feel" the neutral current interaction with other particles
- ν_e feel an additional interaction with electrons in matter
  - strongly affects the mass eigenstates
- ν_e born near the core of the sun have a heavy mass state
  - as they move to lower e- density [n_e], they rotate into ν_μ
  - can convert up to 100% to ν_μ [not limited by $\sin^2 2\theta$]



Sudbury Neutrino Observatory

The ultimate solar neutrino experiment was built in a mine in Sudbury Ontario. It consists of 1000 metric tons of heavy water [D_2O] contained in a 6m radius acrylic sphere viewed by 9600 phototubes. It operated from 1999-2006.

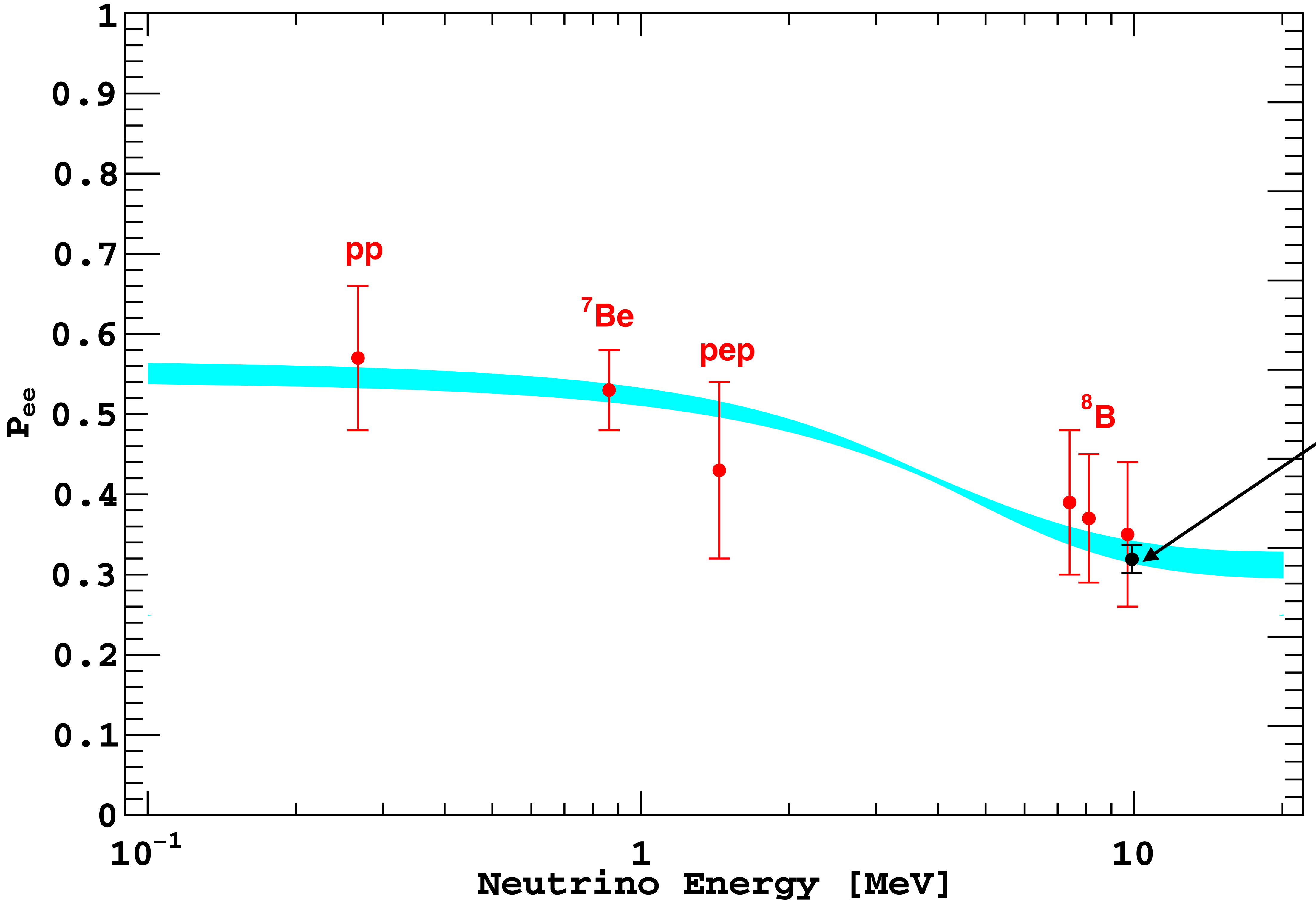


- ^8B neutrinos are detected in 3 reactions:
  - $\nu_e D \rightarrow p\, p\, e^-$: charged current, electron neutrinos only
  - $\nu_x D \rightarrow p\, n\, \nu_x$: neutral current, all neutrino flavors
  - $\nu_x e^- \rightarrow \nu_x e^-$: elastic scattering [mostly ν_e but some others]
- Measured the total rates as follows:
  - $\phi_{cc}/\phi_{nc} = 0.301 \pm 0.033$
  - ϕ_{nc} was completely consistent with the solar model
- The correct rate of neutrinos is observed, only 30% are ν_e

Neutrinos do oscillate!

Solar Neutrino Parameters

A fit of the matter enhanced neutrino oscillation model to all of the data looks like this



SNO Result

the best fit parameters are $\Delta m_{21}^2 = (7.53 \pm 0.18) \times 10^{-5}\ \mathrm{eV}^2$, $\sin^2(2\theta_{12}) = 0.846 \pm 0.021$

Very small square mass difference and very large mixing angle

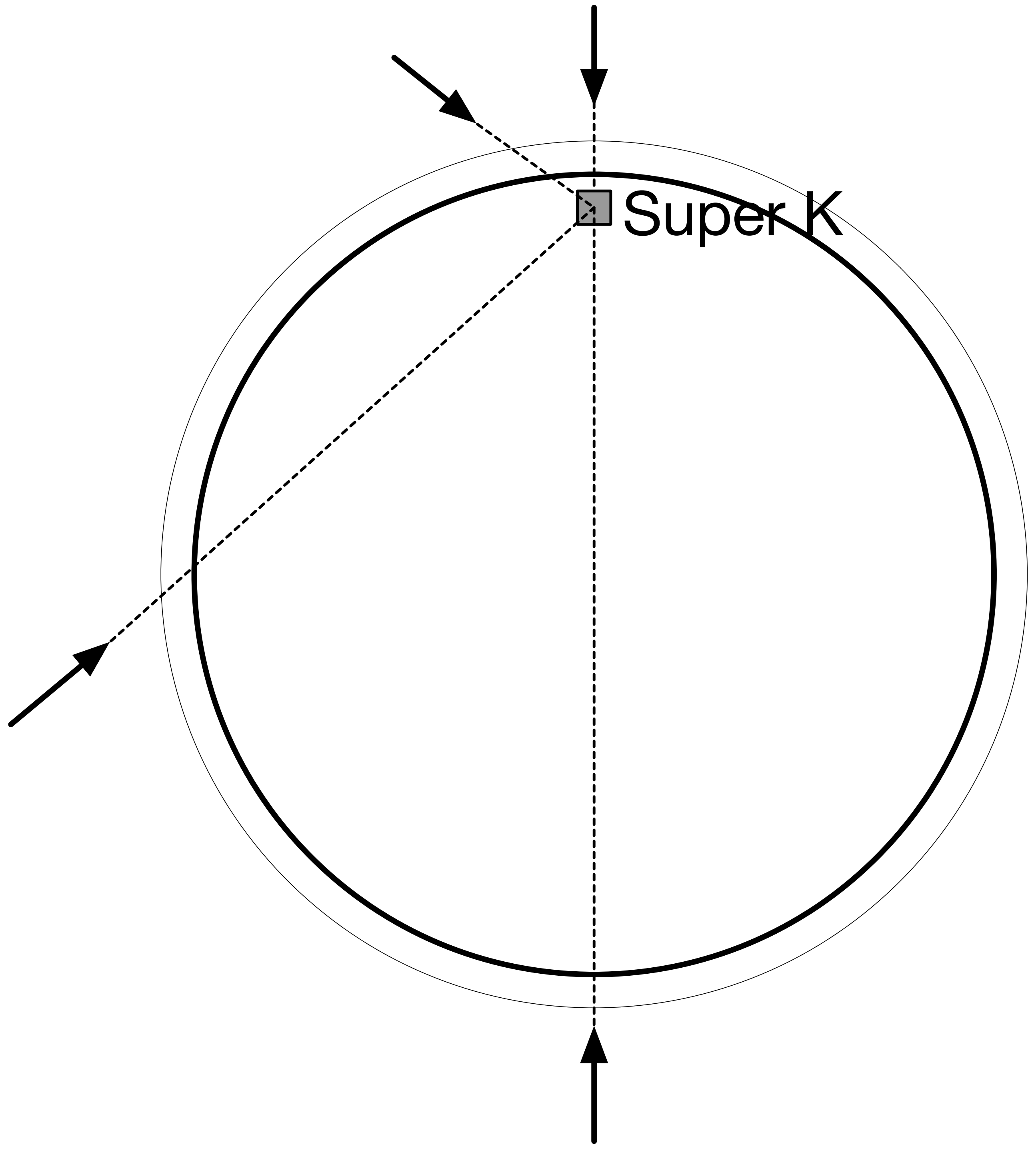
Atmospheric Neutrinos

Cosmic ray showers produce high energy [100 MeV to 1 TeV] ν's. Primary p collide in the upper atmosphere to produce showers of hadrons [mostly pions] which decay to electrons and neutrinos,

$$\pi^- \rightarrow \mu^- \bar{\nu}_\mu \rightarrow e^- \bar{\nu}_e \nu_\mu \bar{\nu}_\mu \qquad \pi^+ \rightarrow \mu^+ \nu_\mu \rightarrow e^+ \nu_e \bar{\nu}_\mu \nu_\mu$$

these impinge on a detector from all directions. There are 2 ν_μ for each ν_e. Because neutrinos easily penetrate the earth, a detector on or near the surface sees ν_x from all directions.

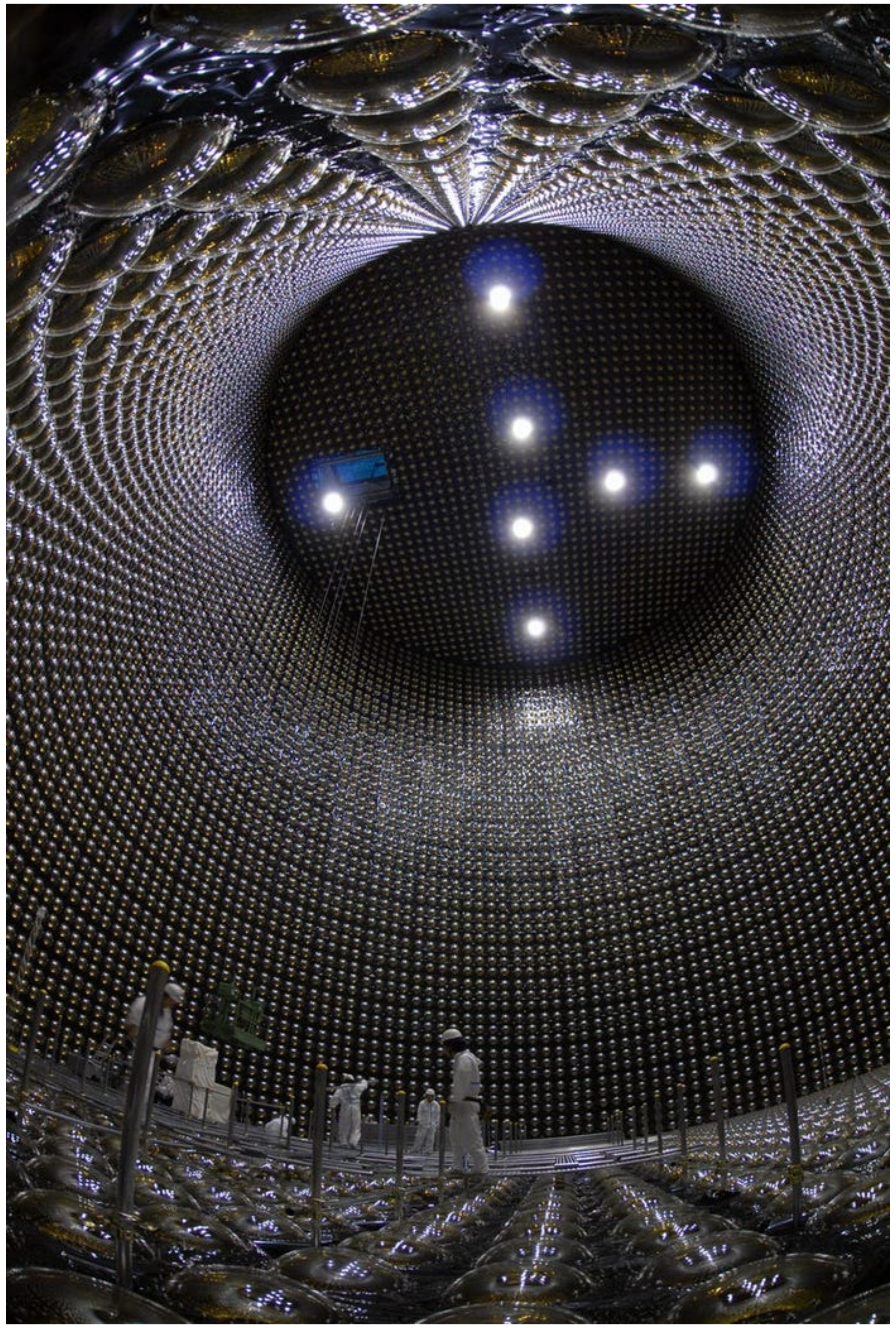
The shortest n path length is at zenith angle $\cos(\theta_z) = 1$, and the longest is at $\cos(\theta_z) = -1$



(Super)Kamiokande

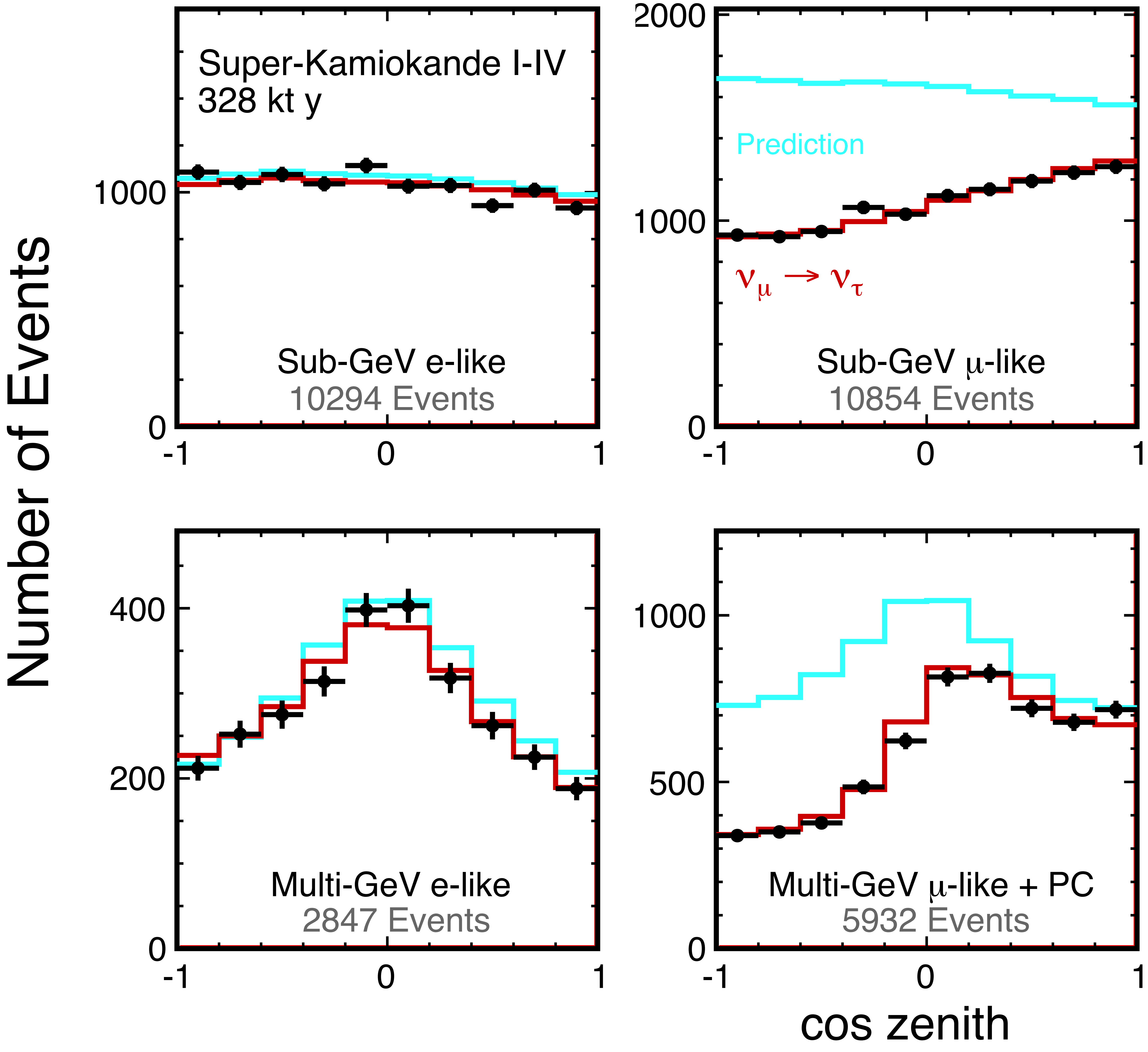
Super Kamiokande is a very large cylindrical water tank of diameter 39m and height 41m. It is filled with 50,000 tons of ultrapure water and instrumented with over 11,000 50cm phototubes.

- Reconstructs the Cerenkov rings from relativistic particles
- It was originally built to search for proton decay p -> $\pi^0 e^+$
  - set the world's best limits
- Used to study solar ν's from $\nu_x e^-$ -> $\nu_x e^-$
- Used to study atmospheric neutrinos
  - can distinguish $e^\pm$ [ragged rings] from $\mu^\pm$ [sharp rings]
  - at the higher energies of atmospheric ν_x interactions, the direction of the final state lepton/initial state ν_x can be measured



The zenith angle distributions of e and μ events are compared with a cosmic ray simulation that does not include any ν oscillations [cyan].

- Events are separated into < 1 GeV and > 1 GeV
- e-like events look exactly as expected
- μ-like events show significant deficits for $\cos\theta_z<1$
  - events with long propagation distances are missing
  - fits to a simple ν oscillation are shown in red
- The ν_x are oscillating and not into ν_e.
  - they are becoming ν_τ



the best fit parameters are $\Delta m_{32}^2 = (2.44\pm0.06)\times10^{-3}$ eV^2, $\sin^2(2\theta_{23})>0.92$ @ 90%CL

Summary

- Neutrinos do indeed have distinct mass eigenstates + the weak flavor eigenstates do mix
- The θ_{12} and θ_{23} mixing angles are large or close to maximal [45°, 45°]
  - very different from the quark weak mixing sector [mixing is 0.2, 0.2^2, 0.2^3]
- If $m_3^2 >> m_2^2 >> m_1^2$
- $m_2 \simeq 8.6\text{x}10^{-3}$ eV, $m_3 \simeq 0.05$ eV
- The m_x are quasi-degenerate then
  - $0.05 \text{ eV} << m_1 \simeq m_2 \simeq m_3 < 1.1 \text{ eV}$
- There are many 21st century long baseline neutrino oscillation experiments that are now contributing to the overall understanding of the 4(6) parameter mixing matrix

Pontecorvo–Maki–Nakagawa–Sakata Matrix

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} .$$

$$U = \underset{\substack{\text{Atmospheric} \\ \boxed{\substack{\mu \to \tau \\ 500\ \text{Km/GeV}}}}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}} \underset{\substack{\text{Reactor/Interference} \\ \boxed{\substack{\mu \leftrightarrow e \\ 500\ \text{Km/GeV}}}}}{\begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}} \underset{\substack{\text{Solar} \\ \boxed{\substack{\mu \to e \\ 15{,}000\ \text{Km/GeV}}}}}{\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$