THE INEVITABILITY OF PHYSICAL LAWS, 1ST ED.

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Poorly translated by J. Smith

Apology:

The universe: once hot and dense and small,

But now a place much larger and chaotic;

With particles and fields still more exotic,

And deep black holes, whose weight will take us all.

And yet, there's hope! For patterns do appear,

Yielding glimpses of the truths most hidden,

In shadows on caves a story's written,

Of symmetries whose reach is far and near.

If in these leaves the errors seem to loom,

Or the notes of illustration seem mere trifles,

Know that you see proof a man was broken:

Blame the scribe, whose glass sees darkly in the gloom,

Whose view of worlds Platonic's wreathed in smoke;

Not the speaker, whose vision has few rivals.

Inevitability of Physical Laws

Nima, 2022

F=ma is almost a tautology (what is force? causes m to a! What's m??)

What else is possible in terms of laws?

Well, you need position, velocity: these are the 2 initial conditions x_0 and v_0, which means they point to a second order differential equation in time

Coulomb force:
Gravity force: $F = \frac{c}{r^2}$ - why not $r^{2.1}$? r^{π}?

In principle, Newtonian mechanics would work fine in any case.

BUT: enter Spacetime & Quantum Mechanics - these lock you in!

one consequence: if both ST and QM, you must have antimatter!

if spin is a thing, and can be $\frac{1}{2}$ & 1, there must be fermions and bosons. (why??)

Dynamics of massless particles: spins can be $\left(0, \frac{1}{2}, 1, \frac{3}{2}, 2\right)$

↳ 10^{-26} seconds ish, because DeBroglie wavelength of heaviest top/Higgs is 10^{-19}

D.F. of massless spin 1 is 2, D.F. of massive spin 1 is 3 (if you can catch up to it, it has 3 arbitrary spin axes)

Claim: the 2 in $\frac{1}{r^2}$ is the same 2 as why the universe is a second-order Diffy q place (???)

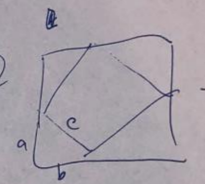
① Pythagoras, rotations & symmetries

② Boosts & relativity

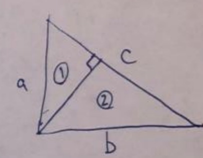
$F=ma$ and $F = \frac{c}{r^2}$ are perfectly compatible with Galilean relativity, but require action-at-a-distance so that 2 things can always reach out and touch each other. If there's a speed limit, the latter isn't possible.

③ Electricity & S.R. → magnetism, accel → radiation

①



Einstein didn't find this, he found a different one.



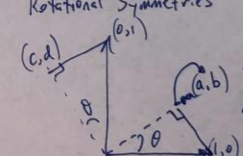
all 3 triangles are similar.

$A_3 = A_1 + A_2$

and area scales with side length^2

so $\left(\text{some constant}\right) c^2 = \left(\text{s.c.}\right) a^2 + \left(\text{s.c.}\right) b^2$

Rotational Symmetries



$\left.\begin{matrix} a^2+b^2=1 \\ c^2+d^2=1 \end{matrix}\right\}$

do it again, (a', b'), (c', d')

$x' = x_0 + ax + by + cx^2 + dy^2 \ldots$

$y' = y_0 + \alpha x + \beta y + \gamma x^2 + \delta y^2 \ldots$

x_0 & y_0 are just translations, so get rid of them.

if $x' \propto cx^2$, then c has units of $\frac{1}{\text{length}}$, which means you don't have a scale-invariant universe! Same for all powers of x, y greater than 1

So $x' = ax + by$ and $y' = \alpha x + \beta y$

rotate by $10°$: $x' = Ax + By$, $y' = Cx + Dy$

rotate by $20°$: just rotate $10°$ twice! This requires rotation invariance, but we have that so cool.

$$x'' = Ax' + By' \quad \text{and} \quad y'' = Cx' + Dy'$$

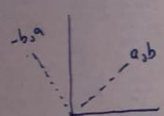
$$x'' = A(Ax+By) + B(Cx+Dy) \qquad y'' = C(Ax+By) + D(Cx+Dy)$$

$$= A^2x + ABy + BCx + BDy \qquad y'' = (AC+CD)x + (BC+D^2y)$$

$$= (A^2+BC)x + (AB+BD)y$$

alt notation: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$A^2 + BC$ etc.



so $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ Ax, By

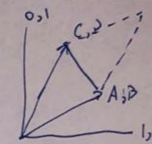
$-Bx, Ay$

let $M_{10°} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ so $\begin{pmatrix} x' \\ y' \end{pmatrix} = M_{10} \begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = M_{10}^2 \begin{pmatrix} x \\ y \end{pmatrix}$

But $M_{10}^{36} \begin{pmatrix} x \\ y \end{pmatrix}$ had better be $\begin{pmatrix} x \\ y \end{pmatrix}$! $(10° \times 36 = 360°!)$

$10°$ was arbitrary so $M_5^{72} = 1$, $M_{90}^4 = 1$, etc.

$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ so



by vector rules, area of isos. triangle (or parallelogram) is $\det M$

and $\det M_1 M_2 = \det M_1 \det M_2$

since $M_{10}^{36} = 1$, and all M_{10} have to be the same thing

so $\det M_{10} = 1$, so $\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = A^2 - (-B)^2 = A^2 + B^2 = 1$

Pythag!

In the same way we can take curves or surfaces and divvy them up into linear dx or square $dx\,dy$ dt, any curved manifold can be approximated into a flat spacetime where $A^2 + B^2 = 1$. Otherwise you have to deal w/ the fact that triangles don't add up to $180°$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \cos^2\theta + \sin^2\theta = 1$ $\quad M(\theta) = \cos\theta \, \mathbb{1} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

I

$M(\theta) = \cos\theta + I \sin\theta$ $\quad 90°$ rotation $= I \cdot I = I^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$??

$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1$

Tiny Rotation$_\varepsilon$ $\quad M(\varepsilon) = 1 + \varepsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad$ let $N\varepsilon = \theta$

$M(N\varepsilon) = (M(\varepsilon))^N$ $\qquad M(\theta) = \left(1 + \frac{\theta}{N} I\right)^N$ as $N \to \infty$

only valid when this is small

$M(\theta) = \left[1 + \frac{\theta}{N} I\right]^N$ as $N \to \infty$

recall $e^x = \lim_{N \to \infty} \left(1 + \frac{1}{N}\right)^N$

so $M(\theta) = e^{I\theta}$

3D rotation $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\substack{xy \\ \text{plane}}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ ignoring z. only rotate in xy plane

$\begin{pmatrix} x' \\ y' \\ z \end{pmatrix}_{yz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & -\sin \\ 0 & \sin & \cos \end{bmatrix}$ $\quad xz = \cancel{\begin{bmatrix} \cos & 0 & -\sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{bmatrix}} \begin{bmatrix} \cos & 0 & -\sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{bmatrix}$

Now, ~~and~~ z-axis rotations aren't the same as x-axis

z then x $\neq$ x then z etc.

$M_z(\varepsilon) = 1 + \varepsilon \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{I_z}$ $\quad I_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad I_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad I_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$R_x(\theta_x) = 1 + \theta_x I_x - \frac{1}{2}\theta_x^2 I_x^2 \ldots$ power series expansion of $e^{I\theta}$

$R_y(\theta_y) = 1 + \theta_y I_y - \frac{1}{2}\theta_y^2 I_y^2$ etc.

$R(\theta_y) R(\theta_x) = (1 + \theta_y I_y)(1 + \theta_x I_x) \ldots$

$\approx 1 + \theta_x I_x + \theta_y I_y + \cancel{\theta_x \theta_y I_x I_y}$ (let θ be small)

$R(\theta_y) R(\theta_x) \approx 1 + \theta_x I_x + \theta_y I_y$

$R(\theta_x) R(\theta_y) \approx 1 + \theta_y I_y + \theta_x I_x$

let's aim for a bit more accuracy

$R_y(\theta_y) R_x(\theta_x) \approx 1 + \theta_x I_x + \theta_y I_y - \frac{1}{2}\theta_y^2 - \frac{1}{2}\theta_x^2 + \theta_y \theta_x I_y I_x$

$R_x \theta_x R_y \theta_y \approx 1 + \theta_y I_y + \theta_x I_x - \frac{1}{2}\theta_x^2 - \frac{1}{2}\theta_y^2 + \theta_x \theta_y I_x I_y$

all the same except $\theta_y \theta_x I_y I_x$ vs. $\theta_x \theta_y I_x I_y$

subtract 2 approximations versions: $R_y \theta_y R_x \theta_x - R_x \theta_x R_y \theta_y = \theta_x \theta_y \left[I_x I_y - I_y I_x\right]$

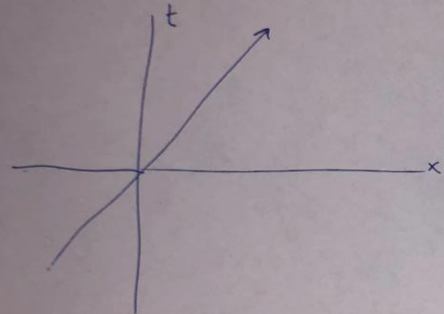
$I_x = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad I_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ $\quad I_x I_y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad I_y I_x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

subtract: $I_x I_y - I_y I_x = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ which is $-I_z$

$I_y I_z - I_z I_y = -I_x \quad , \quad I_z I_x - I_x I_z = -I_y$

② Boosts & Relativity

Start w/ Galileo: $x' = x + vt$; $t' = t$



$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}$ $x' = vt = vt'$

this is already a symmetry that mixes space & time, it just leaves time alone and relates them by a constant.

$x' = A + Bx + B't + \dots$ etc. (swap y for t)

$t' = \alpha + \beta x + \gamma t \dots$ etc.

$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$ Galilean relativity means matrix is $\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

(regenerates $x' = x + vt$, $t' = t$)

$\begin{pmatrix} x' \\ t' \end{pmatrix}_{v_1+v_2} = \begin{bmatrix} 1 & v_1+v_2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{bmatrix} 1 & v_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & v_2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$

$\begin{pmatrix} x' \\ t' \end{pmatrix} = 1 + \varepsilon \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$ thus, $\delta x = \varepsilon t$ and $\delta t = 0$

adding x term to this just rescales x, to that just rescales t

so, try $\delta t = \frac{\varepsilon}{c} x$ ($\frac{1}{c}$ makes units agree)

$x' = x + \delta x$

$t' = t + \delta t$

this means we are now making time change with boost, but not everything is allowed.

No $\delta \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{matrix} x \\ ct \end{matrix}$ for example: violates causality

This gives $\delta x = \varepsilon t$ and $\delta t = \frac{+\varepsilon}{c} x$

$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \left[1 + \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \begin{bmatrix} x \\ ct \end{bmatrix} \Rightarrow \begin{pmatrix} x' \\ ct' \end{pmatrix} = e^{\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{pmatrix} x \\ ct \end{pmatrix}$ (η: rapidity!!; note the +1)

just like $\begin{pmatrix} x' \\ y' \end{pmatrix}_\theta = e^{\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$

we'll call "boost" generator $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$B \cdot B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $e^{\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = 1 + \eta B + \frac{1}{2}\eta^2 B^2 + \frac{1}{3!}\eta^3 B^3 \dots$

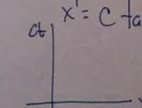
$= \cosh\eta + B\sinh\eta$

Analogy: Rotations $\begin{pmatrix} x' \\ y' \end{pmatrix}_\theta = e^{\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \begin{bmatrix} \cos & -\sin \\ \sin & \cos \end{bmatrix}$

Boosts $\begin{pmatrix} x' \\ ct' \end{pmatrix}_\eta = e^{\eta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{bmatrix} \cosh & \sinh \\ \sinh & \cosh \end{bmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$

$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} 0 \\ ct \end{pmatrix}$ aka particle at rest thru time

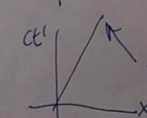
$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{bmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{bmatrix} \begin{pmatrix} 0 \\ ct \end{pmatrix} = \begin{bmatrix} ct & \sinh\eta \\ ct & \cosh\eta \end{bmatrix}$



$x' = c\tanh\eta$

tanh η is bounded by 1, so there's a speed limit - only choice w/ non Galilean boost

Also,



$x' = ct\sinh\eta$

$v = (\tanh\eta)c = \frac{ct'\sinh}{\cosh}$

~~Also~~ Rotations $\begin{pmatrix} x' \\ y' \end{pmatrix} = [\quad] \begin{pmatrix} x \\ y \end{pmatrix}$ Discovery: Pythag $x'^2 + y'^2 = x^2 + y^2$

Boosts $\begin{pmatrix} x' \\ ct' \end{pmatrix} = [\quad] \begin{pmatrix} x \\ ct \end{pmatrix}$ Discovery: $x'^2 - c^2 t'^2 = x^2 - c^2 t^2$

<u>whoa</u>.

Let's do a naive thing and factor.

$x^2 - c^2t^2 \rightarrow (x+ct)(x-ct)$

$x' - ct' = e^{\eta}(x-ct)$

$x' + ct' = e^{\eta}(x+ct)$

$x' = \cosh\eta\, x + \sinh\eta\, t$

$ct' = \cosh\eta\, t + \sinh\eta\, x$

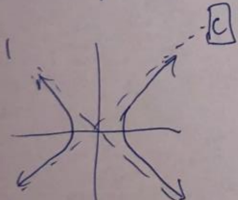
factor $(x^2+y^2) \rightarrow (x+iy)(x-iy)$ comp. conj.

$(x+iy) = z(x+iy)$ $\quad (x-iy) = z^*(x-iy)$ $\quad zz^* = 1$

$x+iy \rightarrow e^{i\theta}(x+iy)$ $\quad x-iy \rightarrow e^{i\theta}(x-iy)$

$x^2 - c^2t^2 = 1$

↑ hyperbola



cos, sin ⇒ rotation

cosh, sinh ⇒ boost

I need to think on this.

Dot Product

$a_x b_x + a_y b_y$ $\quad$ infinitesimal $\delta a_x = \varepsilon a_y$ $\quad \delta b_x = \varepsilon b_y$

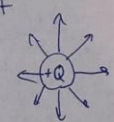
$\delta a_y = -\varepsilon a_x$ $\quad \delta b_y = -\varepsilon b_x$

(Dot products don't change via rotations)

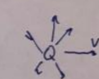
$\delta a_x b_x + a_x \delta b_x + \delta a_y b_y + a_y \delta b_y = \varepsilon a_y b_x + \varepsilon a_x b_y - \varepsilon a_x b_y - \varepsilon a_y b_x + a_x b_x + a_y b_y$

so dot product still works

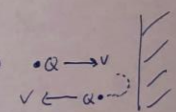
Light



⇒ at low speed,

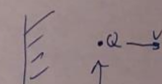


⇒

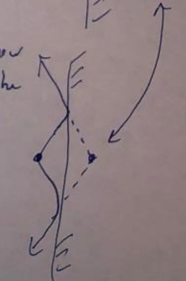


bounces off wall

no wall?



but, way far away, the field lines don't know the bounce happened. so there's a kink on the lines. ~~old~~ outer part

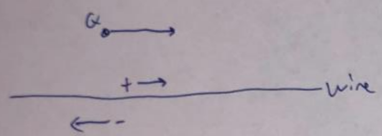


Where is the kink? we claim ct but how to prove it?

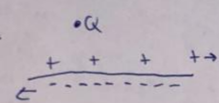
mixups: $\begin{pmatrix}\vec{x}\\ ct\end{pmatrix}$, $\begin{pmatrix}E\\ \vec{p}\end{pmatrix}$, $\begin{pmatrix}\rho\\ \vec{J}\end{pmatrix}$: charge density scalar, current density vector

one transforms to other

slow case:



fast case: put ourselves in α frame:



length contraction of minuses going toward α

so this is electrostatic attraction in α frame

$\begin{pmatrix}ct\\ X\end{pmatrix} \Rightarrow X^{\mu}$ $\quad X \cdot Y = X^0 Y^0 - \vec{x}\cdot\vec{y}$

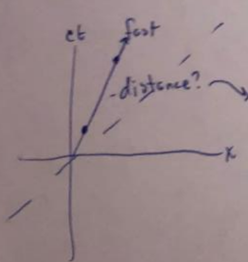
$\mu = 0$ for t

$\mu = 1,2,3$ for spatial dim

$F = ma,\ a = \frac{d^2\vec{x}}{dt^2}$

$c = 1$

$\Delta\tau^2 = \Delta T^2 - \frac{\Delta x^2}{c^2} = \Delta T^2 - \Delta x^2$



$x = vt$

$\Delta t^2 - \Delta x^2 = \Delta t^2(1 - v^2)$

$\frac{d^2 X}{d\tau^2}$ has meaning?

Coulomb force causes accel. : $m\vec{a} = q\vec{E}$

$m\frac{d^2 X}{d\tau^2} = q\cancel{E}$ cannot be right

$d\tau^2 = dt^2 - dx^2$

$\frac{dX}{d\tau}\cdot\frac{dX}{d\tau} = 1$

$d\tau^2 = dX \cdot dX$

$\frac{d}{d\tau}\underbrace{\frac{dx}{d\tau}\cdot\frac{dx}{d\tau}}_{1} = 0$ no good.

define $\mathbb{P} = m\frac{dX}{d\tau}$, $\mathbb{P}\cdot\mathbb{P} = m^2$

I GOT REAL, REAL LOST HERE.

try: $\frac{m d^2 X}{d\tau^2} = \mathbb{R}\left(\mathbb{S}\cdot\frac{dX}{d\tau}\right) - \mathbb{S}\left(\mathbb{R}\cdot\frac{dX}{d\tau}\right)$

$m \cdot \frac{d^2 X}{d\tau^2}\cdot\frac{dX}{d\tau} = \left(\mathbb{R}\frac{dX}{d\tau}\right)\cdot$???