The 'radioactive dice' experiment: why is the 'half-life' slightly wrong?

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Abstract
The 'radioactive dice' experiment is a commonly used classroom analogue to model the decay of radioactive nuclei. However, the value of the half-life obtained from this experiment differs significantly from that calculated for real nuclei decaying exponentially with the same decay constant. This article attempts to explain the discrepancy and suggests modifications to the experiment to minimize this effect.

Introduction

The 'radioactive dice' experiment is commonly performed in schools as an analogy of the decay of radioactive nuclei. A large number of six-sided dice are thrown simultaneously. Those showing a particular number (for example a six) are deemed to have decayed like radioactive nuclei. These dice are removed and the remaining 'undecayed' dice are counted. This number of 'undecayed' dice is recorded and represents the number of undecayed nuclei remaining after a certain interval of time. The 'undecayed' dice are then thrown and, again, those showing a six are removed and the remainder counted. This goes on for a number of throws, resulting in a reduction in the number of 'undecayed' dice as time goes by. Whilst this experiment is meant to represent the decay of radioactive nuclei with a supposed decay constant λ it became clear to the authors over the course of many years of conducting this experiment in the classroom that the 'half-life' (t_{half}) the students obtained experimentally from their dice was consistently lower, on average, than

[1] Retired.

the value predicted using the formula

$$t_{half} = \ln 2/\lambda.$$

In this experiment it is always assumed that because the dice have one chance in six of showing any particular number, then this equates to radioactive decay of real nuclei with a decay constant of $1/6$, since the decay constant is traditionally described as being the probability of any individual nucleus decaying within unit time.

Decay constant
First, though, let us be clear what the radioactive decay constant describes—and it is instructive to realize that it is rather carefully called a decay *constant* not a decay probability or a decay proportion. Depending on the units in which it is measured, its value can be less than, or greater than, one.

How can we have a 'probability' greater than one? Imagine a number of nuclei, with a decay constant of λ s^{-1}, decaying over the course of one second of time, and at the very start of the second the number of undecayed nuclei is N_0. Then the instantaneous activity (in Becquerel) at the start is

Table 1. Number of undecayed dice/nuclei with an initial sample size of 1000.

	Dice decay	Nuclei decay	
Number of mass throws	Number of undecayed dice	Number of undecayed nuclei	Elapsed time (h)
0	1000	1000	0
1	833	846	1
2	694	717	2
3	579	607	3
4	482	513	4
5	402	435	5
6	335	368	6
7	279	311	7
8	233	264	8
9	194	223	9
10	162	189	10
11	135	160	11
12	112	135	12

λN_0. If this activity were maintained throughout the whole of the one second then the number of nuclei that would have decayed by the end of the second is also λN_0.

What actually happens of course is that the activity decreases throughout the second as the number of undecayed nuclei decreases. In the case of long-lived isotopes this decrease should be small and λ is the approximate proportion of the original nuclei that decay in that second. For short-lived isotopes, however, the decrease can be virtually 100% and λN_0 becomes the number that would have decayed in one second if it had been possible to maintain the initial rate of decay for the whole of the second.

Take the example of a radioisotope with $t_{\text{half}} = 13.9$ ms. This time has been chosen to give a nice round number for the decay constant of 50 s^{-1} or, in different units, 0.05 ms^{-1}. Suppose we begin the second with a thousand of this isotope's short-lived nuclei. Then by the end of that second 50 000 of these nuclei would have had time to decay if the high activity at the start of the second could have been maintained—clearly far more than are actually present in the sample. By the same token, 50 would decay in one millisecond if the activity at the start of the millisecond could have been maintained. In fact only 49 (approximately) decay in the millisecond.

So let us now consider two sets of radioactive decay: the 'decay' of a thousand 'radioactive dice' and the decay of a thousand real radioactive nuclei.

Incidentally, we will henceforth keep the 'quotation marks' to a minimum: readers will have probably realized by now that the dice are

not really decaying because they are not *really* radioactive.

The mathematics of the decay of the radioactive dice

Imagine that we were to throw and count the dice at the leisurely rate of once every hour. The simplest way to conduct the experiment is to start with 1000 dice which are all thrown simultaneously at the end of the hour. So one-sixth of them will be removed at the end of the first hour, leaving 833 undecayed if the statistics work perfectly. In general terms the number remaining undecayed after the first simultaneous mass throw is given by

$$N_1 = 1000(1 - 1/6).$$

After two throws the number is now

$$N_2 = 1000(1 - 1/6)(1 - 1/6) = 1000(1-1/6)^2,$$

whilst in general after n mass throws we have

$$N_n = 1000(1 - 1/6)^n.$$

So we see that the decay is given by a geometric progression. The number of undecayed dice obtained using this method for the first 12 throws is shown in table 1.

The mathematics of the decay of real radioactive nuclei

We now need to make a link between the actual elapsed time t used in radioactive decay formulae, and the time interval represented by each mass
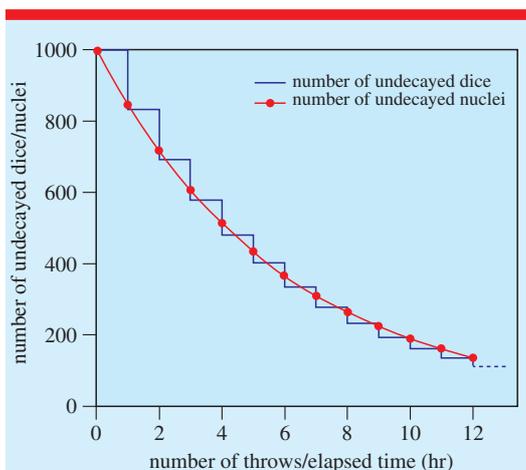
Figure 1. The histogram shows dice decaying as a result of a series of mass throws. The smooth curve is the decay of real nuclei.

throwing of the dice. The decay of real nuclei is given by the exponential equation

$$N_t = N_0 e^{-\lambda t}$$

where N_t is the number of undecayed nuclei remaining after a time t. If we now make the decay constant $1/6 \, h^{-1}$ and let N_0 be equal to 1000 nuclei, this equation becomes

$$N_t = 1000 \, e^{-t/6}$$

and we can count the number of undecayed real nuclei once every hour just like the dice. These values are also shown in table 1 for the first 12 h of decay.

The two decays, a histogram for the dice and a continuous curve for the nuclei, are shown in figure 1.

Half-life of the dice decay

Inspecting the number of undecayed dice in the table tells us that the number halves from 1000 to 500 somewhere between throws 3 and 4. In fact using logarithms tells us that if

$$(1 - 1/6)^n = 0.5 \qquad \text{then}$$

$$n = \ln(0.5)/\ln(1 - 1/6)$$

and thus $n \approx 3.8$.

So after approximately 3.8 'throws' the number of undecayed dice has halved.

Half-life of the nuclei decay

For real nuclei the graph tells us that the half-life is not 3.8 h, but is instead more than 4 h. In fact using

the exponential expression for radioactive decay

$$t_{\text{half}} = \ln 2/\lambda = \ln 2/(1/6)$$

gives a value for t_{half} of 4.16 h, significantly different from the value obtained for the dice.

Why are the two half-lives different?—qualitative approach

Consider the time interval 0–1 h, i.e. the very first hour, of the two decays. The dice only decay when they are thrown at the end of that hour. The number of undecayed dice stays constant at 1000, throughout the hour, and then suddenly drops to 833 at the end of that time, which is why the decay is represented in figure 1 by a histogram.

However, with real nuclei the decay continues throughout the hour. If we start with 1000 undecayed nuclei, we are down to 999 after (approximately) 22 s, to 998 after 44 s, to 997 after 65 s and so on. The decay rate slows as we proceed through the first hour, i.e. as the nuclei decay one after another, the interval between successive decaying nuclei does not have the average value of approximately 22 s: it increases as time passes.

We see that, compared to the dice, fewer nuclei decay in the first hour because the decay rate of the nuclei (the activity) falls over the course of that hour, because the number of undecayed nuclei also falls during this time. The same argument applies for subsequent hours.

A closer analogy?

To bring the dice analogy closer to that of real nuclei, imagine that instead of throwing all 1000 dice at the end of 1 h we do things differently: we throw the dice one at a time, once every 3.6 s, over the course of the first hour: 0–1 h of elapsed time. Some will, when they land, show a six and decay. We note the time interval between the appearances of these sixes. Averaging over the hour we would expect the mean value of this time interval between successive decays to be *constant* at 21.6 s and unlike real nuclei this interval does not increase as we progress through the hour. It is worth pointing out, however, that whilst this may bear more resemblance to the continuous decay of real nuclei, it is *mathematically* identical to throwing all the dice simultaneously at the end of the hour.

Table 2. Comparison of the time interval between successive decays of dice/nuclei.

Number of undecayed dice/nuclei	Time interval between successive 'decays' of dice (s)	Time interval between successive decays of real nuclei (s)
1000	—	—
999	21.611	21.611
998	21.611	21.632
997	21.611	21.654
996	21.611	21.676
995	21.611	21.698

For real nuclei with $t_{half} = 4.16$ h, the increase in the time interval between the decays of the first five nuclei is as shown in table 2.

Continuing with this closer analogy, how do we proceed with the dice over the course of the second hour, from 1 to 2 h of elapsed time? We start this hour with a mean of 833 undecayed dice which need to be thrown regularly over the next 3600 s. We again throw the dice one at a time but now once every 4.32 s. We would now expect sixes to turn up once every 25.93 s on average, again at a constant rate but slower than that of the first hour.

For the third hour the throw rate for the 695 undecayed dice is once every 5.18 s and the 'six rate' will have slowed to one every 31.08 s.

Plotting these results in figure 2 give a series of points, situated at the end of each hour, which are joined by straight lines (indicating a uniform decay rate) whose gradients decrease as the number of hours increases. Compare this with the smooth exponential decay of real nuclei.

As with the simple version of the experiment, the half-life of the dice is still 3.8 (hours).

Why are the two half-lives different?—quantitative approach

In mathematical terms we get different half-lives because the decay of the dice is described by a geometric progression whereas the decay of the nuclei is described by an exponential function. The progression and the function are not identical; however, they are similar initially, so how can we compare the divergence of these two functions? The clearest approach that we have found is to represent each of these functions as a series expansion. For the dice we have the expression

$$N_n = N_0(1 - 1/6)^n$$

where $n = 0, 1, 2, 3, \ldots$ mass throws.
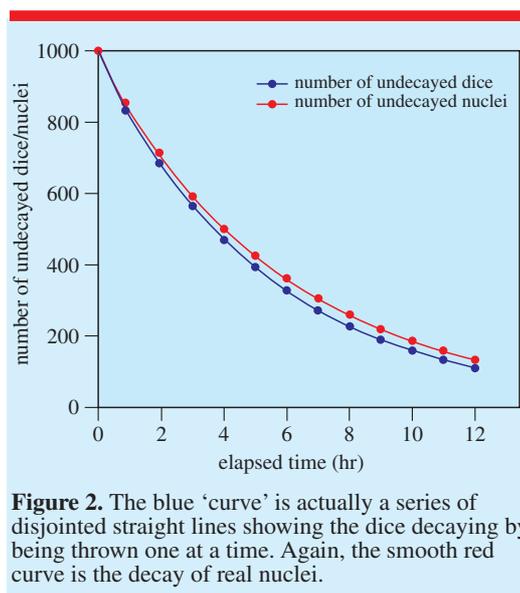
Figure 2. The blue 'curve' is actually a series of disjointed straight lines showing the dice decaying by being thrown one at a time. Again, the smooth red curve is the decay of real nuclei.

Since the general binomial expression $(1+x)^n$ may be represented by the following series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots,$$

we can see that since x in this expression is given by $-1/6$ for our case of six-sided dice, the series expansion for the dice is given by

$$N_n = N_0[1 - \tfrac{1}{6}n + \tfrac{1}{72}n(n-1) - \tfrac{1}{1296}n(n-1)(n-2) + \cdots].$$

In contrast the exponential expression governing the decay of real nuclei is

$$N_t = N_0 e^{-t/6},$$

and since the expansion of the general exponential function e^x is given by

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

we can see that the series expansion for the nuclei is

$$N_t = N_0[1 - \tfrac{1}{6}t + \tfrac{1}{72}t^2 - \tfrac{1}{1296}t^3 + \cdots].$$

It is clear therefore that the two series are very similar in form, but will gradually diverge as either t or n become larger. In fact it is far more instructive to look at these expansions for the more general case in which the dice being used to model the decay have a number of sides p, which would imply a decay constant of $1/p$. In this case it can be shown that the corresponding series expansions for the dice and the nuclei respectively are

$$N_n = N_0 \left[1 - n\left(\frac{1}{p} + \frac{1}{2!p^2} + \frac{2}{3!p^3} + \cdots\right) + n^2\left(\frac{1}{2!p^2} + \frac{3}{3!p^3} + \cdots\right) - \cdots \right]$$

and

$$N_t = N_0\left[1 - t\left(\frac{1}{p}\right) + t^2\left(\frac{1}{2!p^2}\right) - \cdots\right].$$

It is clear from the analysis that the smaller the decay constant, the smaller the divergence, since the additional terms in the expansion of the dice expansion are made up of increasingly higher orders of p. The solution would seem to be to use dice with more sides than six. Such polyhedral dice are used widely in role-playing and war-games, and typically have either 4, 6, 8, 10, 12 or 20 sides. Twenty-sided dice, with a decay constant of 0.05, would give a dice half-life of 13.5 and a nuclei half-life of 13.9—a much closer agreement than that obtained for their six-sided cousins. They are though, nowhere near as cheap as the small plastic or wooden cubes which have one face painted a different colour from the other five and which substitute for dice in the classroom version of the experiment. They would also need to be thrown an inconveniently high number of times to produce useful results.

Conclusion

We have seen that whilst the decay of radioactive nuclei can be modelled reasonably accurately using the throwing of six-sided dice, closer inspection reveals that the half-life obtained using dice is significantly different from that of real nuclei with a decay constant of $1/6$. The fundamental reason for this divergence is that we are modelling a continuous exponential decay using a discrete geometric progression.

We have attempted to explain how the number of sides of the dice affects the degree of divergence and found that it decreases when the number of sides increases. Judging from some worksheets available on the internet, it would seem that a number of people have already come to this conclusion, probably empirically. In practice, though, the commonly available kits (such as those provided by Philip Harris Ltd) work out at a few pence per die, whereas the more unusual polyhedral dice are many times more expensive.

Nevertheless there is no doubt that this simple experiment provides a very useful analogue for radioactive decay—it highlights the random nature of the process, it produces a graph showing the decay rate decreasing with 'number of throws' (or 'time'), and it enables large numbers of pupils to generate their own data simply and safely. We are not suggesting for a moment that teachers should discontinue this experiment, merely to be aware that when the majority of your class find a half-life of around 3.8 when you were expecting 4.2 is not necessarily due to bad luck or a small sample size, but is an inherent limitation of the model.

Dr Arthur Murray retired from teaching in 2007 in order to spend more time with his chickens. He obtained his BSc in physics from the University of Birmingham, UK, in 1965, followed by a PhD in astrophysics from the University of Bristol, UK, in 1969. Periods as a researcher in astrophysics and as a freelance comedy scriptwriter were then followed by teaching physics at St Edward's School, Oxford, UK, and The Oratory School, Woodcote, UK.

Dr Ian Hart led a rather more conventional life. Leaving the University of Bristol, UK, in 1994 with a BSc in physics, a PhD in low temperature physics and a PGCE in secondary science, he then taught physics at St Edward's School, UK, followed by The Oratory School, UK. He is presently Head of Science and Assistant Director of Studies at the latter.